Math 121A: Homework 7 solutions

1. (a) Since $g(x)$ is even, the terms b_n in the Fourier series are zero. The nth cosine term is

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(|x|-\pi)^2\cos(nx)dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}(x-\pi)^2\cos(nx)dx \\
&= -\frac{2}{\pi}\int_{0}^{\pi}2(x-\pi)\frac{\sin(nx)}{n}dx + \left[\frac{2}{\pi}(x-\pi)^2\frac{\sin(nx)}{n}\right]_0^{\pi} \\
&= -\frac{2}{\pi}\int_{0}^{\pi}4\frac{\cos(nx)}{n^2}dx - \left[\frac{2}{\pi}2(x-\pi)\frac{\cos(nx)}{n^2}\right]_0^{\pi} \\
&= -\frac{4}{\pi}\left[\sin(nx)\right]_0^{\pi} + \frac{4\pi}{\pi n^2} \\
&= \frac{4}{n^2}
\end{aligned}$$

and the zeroth cosine term is

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(|x|-\pi)^2dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}(x-\pi)^2dx \\
&= \frac{2}{\pi}\left[\frac{(x-\pi)^3}{3}\right]_0^{\pi} \\
&= \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3}.
\end{aligned}$$

and hence

$$g(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4\cos(nx)}{n^2}.$$

(b) If $f(x,t) = X(x)T(t)$ then

$$T''X = c^2X''T$$

and hence

$$\frac{T''}{c^2T} = \frac{X''}{X} = C$$

for some constant C. Solutions for X that will satisfy the boundary conditions will have $C \leq 0$. First consider the case when $C = 0$. This gives $X'' = 0$ and hence $X(x) = \alpha x + \beta$; in order to satisfy the boundary conditions, it must be that $X(x) = \beta$. The corresponding time dependent problem is $T'' = 0$ and hence $T(t) = \gamma t + \eta$.

The case of $C < 0$ can be considered by defining $C = -\lambda^2$ for $\lambda > 0$. It can be seen that

$$X'' + \lambda^2 X = 0$$

and hence

$$X(x) = D\cos\lambda x + E\sin\lambda x.$$

The derivative is

$$X'(x) = -D\lambda\sin\lambda x + E\lambda\cos\lambda x.$$

The boundary conditions imply that

$$0 = X'(\pi) = -D\lambda\sin\lambda\pi + E\lambda\cos\lambda\pi,$$

$$0 = X'(-\pi) = -D\lambda\sin\lambda\pi + E\lambda\cos\lambda\pi.$$

There are two families of solutions. If $E = 0$ and $D \neq 0$ then the boundary conditions will be satisfied if $\lambda = n$ where n is a positive integer. Thus $X(x) = \cos(nx)$, and the corresponding time-dependent problem, given by $C = -n^2$, is

$$\frac{T''}{c^2 T} = -n^2.$$

Therefore $T(t) = A\cos nct + B\sin nct$. If $D = 0$ and $E \neq 0$ then the boundary conditions will be satisfied if $\lambda = (n + 1/2)$. The corresponding time dependent solution will be $T(t) = A\cos(n + 1/2)ct + B\sin(n + 1/2)ct$.

(c) In part (a), the initial condition was written as a sum of terms of the form $\cos(nx)$, plus a constant term—these are the spatial parts of separable solutions. Since f_t is initially zero, it follows that the solution for all times is a sum of separable solutions,

$$f(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4\cos(nx)\cos(nct)}{n^2}.$$

(d) The energy of the system can be written as

$$E(t) = \frac{1}{2}\int_{-\pi}^{\pi} (f_t^2 + c^2 f_x^2)dx.$$

The derivative of this expression is

$$\begin{aligned}
E'(t) &= \frac{1}{2}\frac{d}{dt}\left(\int_{-\pi}^{\pi}(f_t^2 + c^2 f_x^2)dx\right) \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\frac{\partial}{\partial t}(f_t^2 + c^2 f_x^2)dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi}(2f_t f_{tt} + 2c^2 f_{xt} f_x)dx \\
&= \int_{-\pi}^{\pi} f_t f_{tt} dx + c^2\int_{-\pi}^{\pi} f_{xt} f_x dx \\
&= \int_{-\pi}^{\pi} f_t f_{tt} dx - c^2\int_{-\pi}^{\pi} f_t f_{xx} dx + [f_t f_x]_{-\pi}^{\pi} \\
&= \int_{-\pi}^{\pi} f_t (f_{tt} - c^2 f_{xx}) dx \\
&= \int_{-\pi}^{\pi} 0\, dx \\
&= 0
\end{aligned}$$

and hence the total energy is constant.

(e) Since f_t is initially zero, it follows that $K(0) = 0$. For $x > 0$,

$$\left.\frac{\partial f}{\partial x}\right|_{t=0} = 2(x - \pi)$$

and hence

$$\begin{aligned}
P(0) &= \frac{1}{2}\int_{-\pi}^{\pi} c^2 f_x^2 dx \\
&= c^2\int_{0}^{\pi} 4(x-\pi)^2 dx \\
&= \frac{4c^2\pi^3}{3}.
\end{aligned}$$

(f) The time derivative of the series solution is

$$f_t(x,t) = -\sum_{n=1}^{\infty}\frac{4c\cos(nx)\sin(nct)}{n}$$

and hence the kinetic energy is

$$\begin{aligned}
K(t) &= \frac{1}{2}\int_{-\pi}^{\pi} f_t^2 dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty}\frac{4c\cos(nx)\sin(nct)}{n}\right)^2 dx.
\end{aligned}$$

By orthogonality relations, all integrals involving $\cos(nx)\cos(mx)$ for $m \neq n$ will vanish and hence

$$\begin{aligned} K(t) &= 8c^2 \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \frac{\cos^2(nx)\sin^2(nct)}{n^2} dx \\ &= 8c^2 \pi \sum_{n=1}^{\infty} \frac{\sin^2(nct)}{n^2}. \end{aligned}$$

The spatial derivative of the series solution is

$$f_x(x,t) = -\sum_{n=1}^{\infty} \frac{4\sin(nx)\cos(nct)}{n}$$

and hence the potential energy is

$$\begin{aligned} P(t) &= \frac{1}{2}\int_{-\pi}^{\pi} c^2 f_x^2 dx \\ &= \frac{c^2}{2}\int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} \frac{4\sin(nx)\cos(nct)}{n} \right)^2 dx \\ &= 8c^2 \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \frac{\sin^2(nx)\cos^2(nct)}{n^2} \\ &= 8c^2 \pi \sum_{n=1}^{\infty} \frac{\cos^2(nct)}{n^2}. \end{aligned}$$

Hence, using the relation $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$,

$$\begin{aligned} E(t) &= K(t) + P(t) \\ &= 8c^2\pi \sum_{n=1}^{\infty} \frac{\cos^2(nct) + \sin^2(nct)}{n^2} \\ &= 8c^2\pi \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= 8c^2\pi \left(\frac{\pi^2}{6} \right) \\ &= \frac{4c^2\pi^3}{3}, \end{aligned}$$

which is constant in time, and matches the calcaultion in part (e).

(g) Plots of $f(x,t)$ are shown over the range from $t = 0$ to $t = \pi/c$ in Fig. 1. After $t = \pi/c$, the waves reverse, and follow the same sequence of curves back to the initial condition, at time $t = 2\pi/c$.

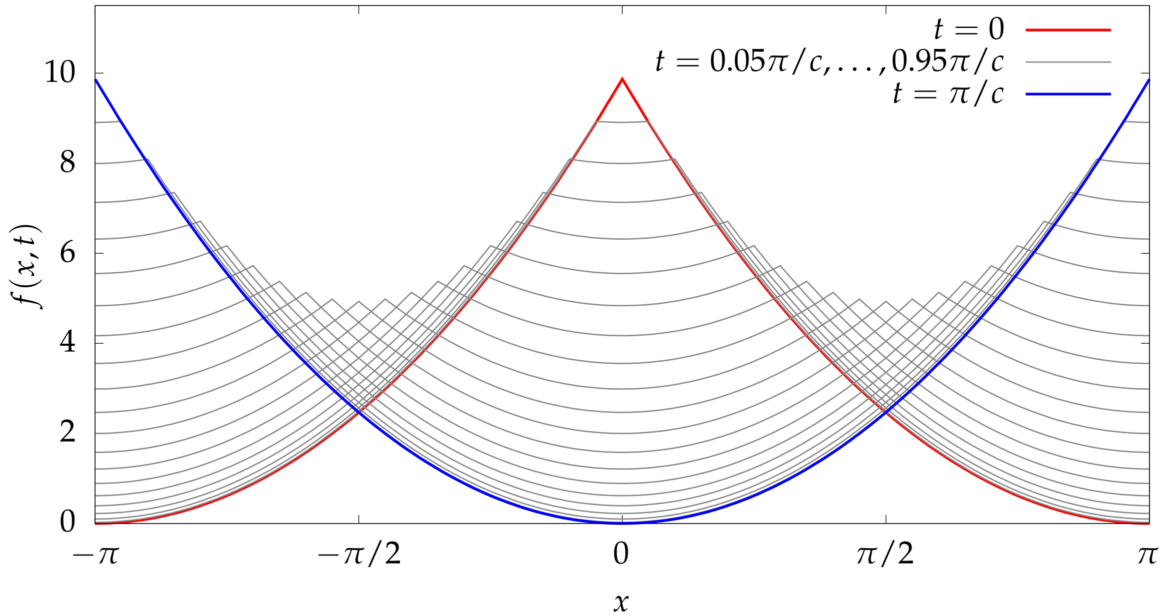

Figure 1: Time evolution of the function $f(x,t)$ considered in question 1.

2. (a) The Fourier transform is given by

$$
\begin{aligned}
\tilde{f}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\alpha}dx \\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (\cos x)e^{-ix\alpha}dx \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} (e^{ix}+e^{-ix})e^{-ix\alpha}dx \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} (e^{ix(1-\alpha)}+e^{-ix(1+\alpha)})dx \\
&= \frac{1}{4\pi}\left[\frac{e^{ix(1-\alpha)}}{i(1-\alpha)} - \frac{e^{-ix(1+\alpha)}}{i(1+\alpha)}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{4\pi}\left[\frac{ie^{-i\pi\alpha/2}}{i(1-\alpha)} + \frac{ie^{-i\pi\alpha/2}}{i(1+\alpha)} + \frac{ie^{i\pi\alpha/2}}{i(1-\alpha)} + \frac{ie^{-i\pi\alpha/2}}{i(1+\alpha)}\right] \\
&= \frac{1}{2\pi}\left[\frac{\cos\left(\frac{\pi\alpha}{2}\right)}{1-\alpha} + \frac{\cos\left(\frac{\pi\alpha}{2}\right)}{1+\alpha}\right] \\
&= \frac{\cos\left(\frac{\pi\alpha}{2}\right)}{\pi(1-\alpha^2)}.
\end{aligned}
$$

(b) The Fourier transform is

$$
\begin{aligned}
\tilde{g}(\alpha) &= \frac{1}{2\pi}\int_0^{\pi}(\sin x)e^{-ix\alpha}dx \\
&= \frac{1}{4\pi i}\int_0^{\pi}(e^{ix(1-\alpha)} - e^{-ix(1+\alpha)})dx \\
&= \frac{1}{4\pi i}\left[\frac{e^{ix(1-\alpha)}}{i(1-\alpha)} + \frac{e^{-ix(1+\alpha)}}{i(1+\alpha)}\right]_0^{\pi} \\
&= \frac{1}{4\pi i}\left[\frac{-1-e^{i\alpha\pi}}{i(1-\alpha)} + \frac{-1-e^{i\alpha\pi}}{i(1-\alpha)}\right] \\
&= \frac{1+e^{-i\pi\alpha}}{2\pi(1-\alpha^2)}.
\end{aligned}
$$

(c) $\tilde{g}$ can be alternatively written as

$$
\tilde{g}(\alpha) = \frac{e^{-i\pi\alpha/2}(e^{i\pi\alpha/2} + e^{-i\pi\alpha/2})}{2\pi(1-\alpha^2)} = \frac{e^{-i\pi\alpha/2}\cos\left(\frac{\pi\alpha}{2}\right)}{\pi(1-\alpha^2)}
$$

and hence

$$
\frac{\tilde{f}(\alpha)}{\tilde{g}(\alpha)} = e^{i\pi\alpha/2}.
$$

This should be expected. Since $\cos x = \sin(x + \pi/2)$, it can be seen that $f(x) = g(x + \pi/2)$ and by basic properties of Fourier transforms, $\tilde{f}(\alpha) = e^{i\alpha\pi/2}\tilde{g}(\alpha)$.

(d) To calculate the Fourier transform of h, first note that

$$
h(x) = g(x) - g(-x).
$$

The Fourier transform of $-g(-x)$ is $-\tilde{g}(-\alpha)$, and hence

$$
\begin{aligned}
\tilde{h}(\alpha) &= \tilde{g}(\alpha) - \tilde{g}(-\alpha) \\
&= \frac{1+e^{-i\pi\alpha}}{2(1-\alpha^2)} - \frac{1+e^{i\pi\alpha}}{2(1-(-\alpha)^2)} \\
&= \frac{e^{-i\pi\alpha} - e^{-i\pi\alpha}}{2(1-\alpha^2)} \\
&= \frac{i\sin(\pi\alpha)}{\pi(\alpha^2-1)}.
\end{aligned}
$$

(e) The function $q_n(x)$ can be written as

$$
q_n(x) = (-1)^{n-1}\sum_{k=0}^{n-1} h(x - (2k-(n-1))\pi)
$$

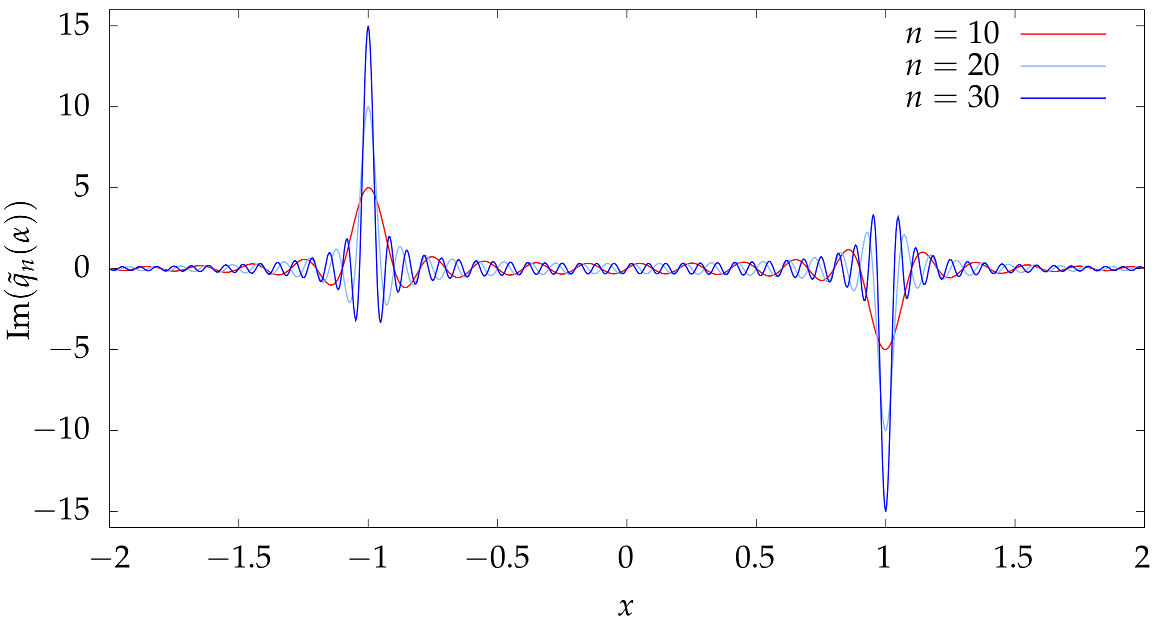

Figure 2: Fourier transforms of several of the truncated sine functions $q_n(x)$ that are considered in question 2.

and hence

$$\begin{aligned}
\tilde{q}_n(\alpha) &= (-1)^{n-1}\sum_{k=0}^{n-1} e^{-i(2k-(n-1))\pi\alpha}\tilde{h}(\alpha) \\
&= (-1)^{n-1}\tilde{h}(\alpha)e^{-i\pi(n-1)\alpha}\sum_{k=0}^{n-1} e^{2ki\pi\alpha} \\
&= (-1)^{n-1}\left(\frac{i\sin(\pi\alpha)}{\pi(\alpha^2-1)}\right)e^{-i\pi(n-1)\alpha}\frac{1-e^{2ni\pi\alpha}}{1-e^{2i\pi\alpha}} \\
&= (-1)^{n}\left(\frac{i\sin(\pi\alpha)}{\pi(1-\alpha^2)}\right)\frac{e^{-ni\pi\alpha}-e^{ni\pi\alpha}}{e^{-i\pi\alpha}-e^{i\pi\alpha}} \\
&= (-1)^{n}\left(\frac{i\sin(\pi\alpha)}{\pi(1-\alpha^2)}\right)\frac{\sin(n\pi\alpha)}{\sin(\pi\alpha)} \\
&= \frac{i(-1)^n\sin(n\pi\alpha)}{\pi(1-\alpha^2)}.
\end{aligned}$$

Plots of $\tilde{q}_n(\alpha)$ for $n = 10, 20, 30$ are shown in Fig. 2. It can be seen that there are two sharp peaks in $\tilde{q}_n(\alpha)$ at $\alpha = \pm 1$, which grow in size as n increases. This should be expected, since in the limit as n tends to infinity, $q_n(x)$ becomes equal to $\sin(nx) = \frac{1}{2i}(e^{ix} - e^{-ix})$. Since the Fourier transform of e^{ikx} can be thought of as $\delta(\alpha - k)$, it should be expected that the limit of $\tilde{q}_n(\alpha)$ is

$$\frac{1}{2i}(\delta(\alpha-1)-\delta(\alpha+1)) = \frac{i}{2}(\delta(\alpha+1)-\delta(\alpha-1))$$

which matches the peaks seen in the graph.

3. If the rate of change of the temperature of the tea is given by λ multiplied by the difference between the tea's temperature and T_r, then the temperature $T(t)$ will follow the differential equation

$$\frac{dT}{dt} = \lambda(T_r - T).$$

To solve this equation, it can be separated according to

$$\frac{dT}{T - T_r} = -\lambda dt$$

and hence

$$\log(T - T_r) = -\lambda t + C$$

for some constant C. Therefore if $T(0) = T_0$ the temperature evolves according to

$$T(t) = T_r + (T_0 - T_r)e^{-\lambda t}.$$

Now consider the first scenario where the milk is added initially. The initial temperature will be given by the weighted average of the temperatures of the tea and milk,

$$T_0 = \frac{T_t V_t + T_m V_m}{V_t + V_m} = \frac{(95\ ^\circ\mathrm{C})200 + (5\ ^\circ\mathrm{C})50}{250} = 77\ ^\circ\mathrm{C}.$$

The temperature then evolves according to

$$T(t) = (20 + 57e^{-\lambda t})\ ^\circ\mathrm{C}$$

and thus after twenty minutes it will be

$$T(20 \text{ min}) = (20 + 57e^{-4\log(2)})\ ^\circ\mathrm{C} = 23.56\ ^\circ\mathrm{C}.$$

In the second scenario, the initial temperature is 95 $^\circ$C and it evolves according to

$$T(t) = (20 + 75e^{-\lambda t})\ ^\circ\mathrm{C}$$

so after twenty minutes it will be

$$T(20 \text{ min}) = (20 + 75e^{-4\log(2)})\ ^\circ\mathrm{C} = 24.69\ ^\circ\mathrm{C}.$$

After the milk is added, the temperature is then

$$\frac{(24.69\ ^\circ\mathrm{C})200 + (5\ ^\circ\mathrm{C})50}{250} = 20.75\ ^\circ\mathrm{C}$$

Hence the tea is hotter under the first scenario.

4. Consider a function $y(x)$. If $y^2 = r^2 - x^2$, then $yy' = rr' - x$. Multiplying the given differential by y gives

$$y^2 y'^2 + 2xyy' - y^2 = 0$$

and substituting the expressions for r gives

$$(rr' - x)^2 + 2x(rr' - x) - (r^2 - x^2) = 0.$$

Expanding terms gives

$$r^2 r'^2 - 2xrr' + x^2 + 2xrr' - 2x^2 - r^2 + x^2 = 0$$

and hence

$$r^2 r'^2 - r^2 = 0.$$

Assuming that $r > 0$ gives

$$r' = \pm 1, \qquad r(x) = \pm x + C$$

and hence

$$y(x) = \sqrt{(\pm x + C)^2 - x^2} = \sqrt{C^2 \pm 2xC}.$$

5. For the system considered,

$$\tan(2\theta) = \frac{y}{x}, \qquad \tan\theta = \frac{dy}{dx}$$

and by using the trigonometric identity

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

it follows that

$$\frac{y}{x} = \frac{2y'}{1-y'^2}.$$

Thus

$$(1-y'^2)y = 2y'x$$

and hence

$$y'^2y + 2xy' - y = 0$$

This is the equation that was considered in the previous question, and hence

$$y(x) = \sqrt{C^2 \pm 2xC},$$

which is a parabola.