Math 121A: Homework 5 solutions

1. (a) The volume of the pyramid is

$$\begin{aligned} V &= \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \\ &= \int_0^1 dz \int_0^{1-z} (1-y-z)dy \\ &= \int_0^1 dz \left[(1-z)y - y^2\right]_0^1 - z \\ &= \int_0^1 \frac{(1-z)^2}{2} dz \\ &= \int_0^1 \frac{w^2}{2} dw = \frac{1}{6}. \end{aligned}$$

(b) To calculate the z coordinate of the centroid, the same calculation as above can be employed but with an addition factor of z. Hence

$$\begin{aligned} V\bar{z} &= \int_0^1 z\frac{(1-z)^2}{2} dz \\ &= \frac{1}{2}\int_0^1 (z - 2z^2 + z^3) dz \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\ &= \frac{1}{24} \end{aligned}$$

and therefore $\bar{z} = 1/4$. To find the other coordinates of the centroid, note that the pyramid is identical if any pair of coordinate axes are switched, and hence it must be that $\bar{x} = \bar{y} = \bar{z} = 1/4$.

(c) If the density of the object is z, then its mass is

$$M = \int_0^1 z\frac{(1-z)^2}{2} dz,$$

which is exactly the same integral as in part (b), and hence evaluates to $1/24$. The z coordinate of the center of mass is given by

$$\begin{aligned} M\bar{z} &= \int_0^1 z^2\frac{(1-z)^2}{2} dz \\ &= \frac{1}{2}\int_0^1 (z^2 - 2z^3 + z^4) dz \\ &= \frac{1}{2}\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) \\ &= \frac{1}{60}. \end{aligned}$$

Hence

$$\bar{z} = \frac{M\bar{z}}{M} = \frac{24}{60} = \frac{2}{5}.$$

2. By using the spherical coordinate system (r, θ, ϕ), the volume is given by

$$\begin{aligned} V &= \int_0^a dr \int_0^\alpha d\theta \int_0^{2\pi} d\phi \, r^2 \sin\theta \\ &= \left(\int_0^a r^2 \, dr\right) \left(\int_0^\alpha \sin\theta \, d\theta\right) 2\pi \\ &= \frac{2\pi a^3}{3} \left[-\cos\theta\right]_0^\alpha \\ &= \frac{2\pi a^3(1-\cos\alpha)}{3}. \end{aligned}$$

The moment in the z direction is given by

$$\begin{aligned} V\bar{z} &= \int_0^a dr \int_0^\alpha d\theta \int_0^{2\pi} d\phi \, (r\cos\theta) r^2 \sin\theta \\ &= \left(\int_0^a r^3 \, dr\right) \left(\int_0^\alpha \frac{\sin 2\theta}{2} \, d\theta\right) 2\pi \\ &= \frac{\pi a^4}{8} \left[-\cos 2\theta\right]_0^\alpha \\ &= \frac{\pi a^4}{8}(1-\cos 2\alpha) \\ &= \frac{\pi a^4}{8}(2 - 2\cos^2\alpha) \\ &= \frac{\pi a^4}{4}(1-\cos\alpha)(1+\cos\alpha) \end{aligned}$$

and hence

$$\bar{z} = \frac{\pi a^4(1-\cos\alpha)(1+\cos\alpha)3}{4(2\pi a^3(1-\cos\alpha))} = \frac{3a(1+\cos\alpha)}{8}.$$

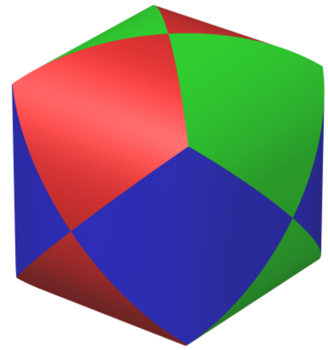
3. For the parabolic cylinder coordinate system where $x = (u^2 - v^2)/2$ and $y = uv$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2.$$

4. (a) The object is plotted in Fig. 1(a). To evaluate its volume, note that the cross section of the object through a plane of constant x is a square, given by

$$|y| \le \sqrt{1-x^2}, \qquad |z| \le \sqrt{1-x^2}.$$

(a)



(b)

Figure 1: Objects considered in question 4. In (a) the volume formed by the intersection between $x^2 + y^2 \leq 1$ (shown in blue) and $x^2 + z^2 \leq 1$ (shown in green) is plotted. In (b) the additional constraint of $y^2 + z^2 \leq 1$ (shown in red) is considered. (Images made with POV-Ray (`www.povray.org`).)

The volume can therefore be integrated as

$$V = \int_{-1}^{1} 4(1 - x^2)dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 8\left(1 - \frac{1}{3}\right) = \frac{16}{3} = 5.333.$$

(b) The object is plotted in Fig. 1(b). There are many ways to evaluate its volume, although one way is to note that it can be broken down into a cube $|x| \leq 1/\sqrt{2}$, $|y| \leq 1/\sqrt{2}$, $|z| \leq 1/\sqrt{2}$, and six identical volume patches, attached to each face of the cube. Each patch has a volume that can be evaluated using a similar integral to part (a):

$$\begin{aligned} V_s = \int_{1/\sqrt{2}}^{1} 4(1 - x^2)dx &= 4\left[x - \frac{x^3}{3}\right]_{1/\sqrt{2}}^{1} \\ &= 4\left[\frac{2}{3} - \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right] \\ &= \frac{8}{3} - \frac{5\sqrt{2}}{3}. \end{aligned}$$

Hence the volume of the object is

$$V = 2\sqrt{2} + 6V_s = 16 - 8\sqrt{2} = 4.686.$$

5. The equation can be expanded as

$$-e^{-2x}\frac{d^2y}{dx^2} + 2xe^{-2x}\frac{dy}{dx} - e^{-2x}y = \lambda e^{-2x}y,$$

which can be simplified to

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + (1+\lambda) = 0.$$

Searching for solutions of the form e^{pt} gives

$$p^2 - 2p + 1 + \lambda = 0, \qquad (p-1)^2 = -\lambda.$$

If $\lambda < 0$, so that it can be written as $\lambda = -\mu^2$ for $\mu > 0$, then $p = 1 \pm \mu$. This can be written as

$$y(x) = Ae^{(1+\mu)x} + Be^{(1-\mu)x}$$

for some constants A and B. This an equivalently be parameterized by

$$y(x) = e^x(C\cosh\mu + D\sinh\mu)$$

for some constants C and D. Since $y(0) = 0$, it follows that $C = 0$. Hence if $y(x) = De^x \sinh\mu$ then

$$y'(x) = De^x(\sinh\mu x + \mu\cosh\mu x)$$

and hence

$$y'(\pi) - y(\pi) = D\mu\cosh\mu\pi$$

which will never be zero unless $D = 0$. Hence there are no eigenfunctions for $\lambda < 0$. Similar considerations can rule out the case when $\lambda = 0$. If $\lambda > 0$, so that $\lambda = q^2$ for $q > 0$, then

$$y(x) = e^x(A\sin qx + B\cos qx)$$

The condition $y(0) = 0$ implies $B = 0$, and hence $y(x) = Ae^x \sin qx$, so

$$y'(0) = Ae^x(\sin qx + q\cos qx).$$

The other boundary condition gives

$$y'(\pi) - y(\pi) = Aq\cos q\pi.$$

This will be zero for $q = (n - 1/2)$ where $n = 1, 2, 3, \ldots$. Thus the eigenvalues are $\lambda_n = (n - 1/2)^2$ with corresponding eigenfunctions

$$y_n(x) = e^x \sin(n - 1/2)x$$

To verify orthogonality, consider two eigenfunctions y_m and y_n where $m \neq n$. The innner product is

$$\begin{aligned}
\langle y_m, y_n \rangle &= \int_0^\pi w(x) y_m(x) y_n(x) dx \\
&= \int_0^\pi e^{-2x} (e^x \sin[(m - 1/2)x])(e^x \sin[(n - 1/2)x]) dx \\
&= \int_0^\pi \sin[(m - 1/2)]x \sin[(n - 1/2]) x dx \\
&= \int_0^\pi \frac{\cos[(m-n)x] - \cos[(m+n+1)x]}{2} dx \\
&= \frac{1}{2} \left[\frac{\sin[(m-n)x]}{m-n} - \frac{\sin[(m+n+1)x]}{m+n+1} \right]_0^\pi \\
&= 0
\end{aligned}$$

since $\sin k\pi = 0$ for any integer k.

6. (a) Substituting $f(x,t) = X(x)T(t)$ into the equation gives

$$XT' = bX''T$$

which can be rearranged to give

$$\frac{T'}{bT} = \frac{X''}{X}.$$

Since the LHS is a function of t only, and the RHS is a function of x only, it follows that both sides are equal to some constant C. The function X therefore satisfies

$$X'' = CX$$

with the conditions $X(0) = 0$ and $X'(a) = 0$, making it a Sturm–Liouville problem. If $C > 0$ then the solutions can be written as

$$X(x) = A \cosh qx + B \sinh qx$$

for $q = \sqrt{C}$, where A and B are arbitrary constants. Since $X(0) = 0$ implies $A = 0$, and $X'(a) = 0$ implies $B = 0$, it follows that there are no non-zero solutions of this form that satisfy the boundary conditions. Similarly, if $C = 0$ then $X(x) = Ax + B$, and there are no non-zero solutions that satisfy the boundary conditions. If $C < 0$, then

$$X(x) = A \cos qx + B \sin qx$$

for $q = \sqrt{-C}$, where A and B are arbitrary constants. The condition $X(0) = 0$ gives $A = 0$, and the condition $X'(a) = 0$ implies

$$0 = \frac{1}{q} \cos qa$$

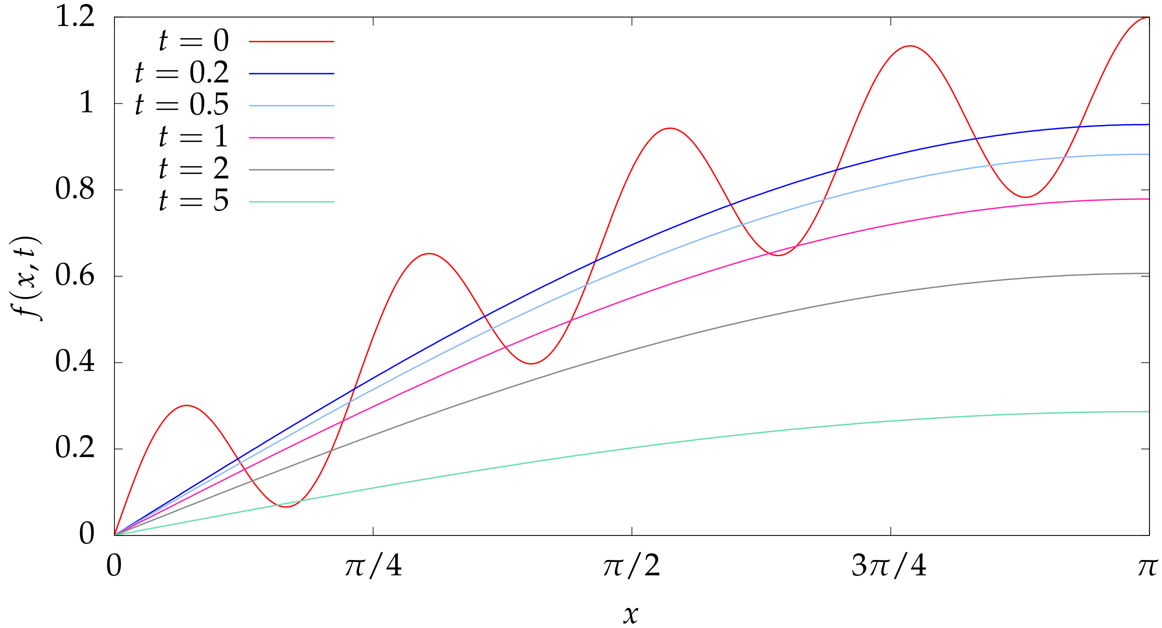


Figure 2: Plots of the concentration $f(x,t)$ of gas in a channel following the diffusion equation, for several different time points.

for non-zero solutions, and hence $q = \pi(n - 1/2)/a$ for $n = 1, 2, 3, \ldots$. The equation for the time dependence is then

$$T' = -bq^2 T = -b\left(\frac{\pi(n-1/2)}{a}\right)^2 T.$$

and hence

$$T(t) = D\exp\left(-\frac{b(n-1/2)^2\pi^2 t}{a^2}\right)$$

for some constant D. Hence separable solutions have the form

$$\sin\left(\frac{\pi(n-1/2)t}{a}\right)\exp\left(-\frac{b(n-1/2)^2\pi^2 t}{a^2}\right).$$

(b) Since the initial condition comprises of a linear combination of the separable solutions found in part (a), the solution will be given by those separable solutions, and hence

$$f(x,t) = \sin\frac{\pi x}{2a}\exp\left(-\frac{b\pi^2 t}{4a^2}\right) + \frac{1}{5}\sin\frac{17\pi x}{2a}\exp\left(-\frac{289b^2\pi^2 t}{4a^2}\right).$$

If $a = \pi$ and $b = 1$ this simplifies to

$$f(x,t) = e^{-t/4} \sin \frac{x}{2} + \frac{e^{-289t/4}}{5} \sin \frac{17\pi x}{2}.$$

Figure 2 shows plots of $f(x,t)$ for the values of t of 0, 0.2, 0.5, 1.0, 2.0, and 5.0. It can be seen that the high frequency oscillations are damped much more rapidly and are barely visible at $t = 0.2$. This should be expected, since the expontential for this term has a significantly faster rate of decay. For later times, the low frequency component of the gas distribution begins to decay also, as the gas is removed at $x = 0$.