Math 121A: Homework 4 solutions

1. The plane can be written as $\mathbf{r} \cdot \mathbf{n} = 10$ where $\mathbf{n} = (2, 6, -3)$. The length of $\mathbf{n}$ is
$$|\mathbf{n}| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7.$$
Hence a unit normal can be written as $\hat{\mathbf{n}} = \mathbf{n}/7$ and thus the plane can be written as $\mathbf{r} \cdot \hat{\mathbf{n}} = {}^{10}/_{7}$. In this form, the quantity $\mathbf{r} \cdot \hat{\mathbf{n}}$ represents the distance of $\mathbf{r}$ in the direction of $\hat{\mathbf{n}}$.
For the position $\mathbf{x} = (-2, 4, 5)$, its distance from the plane is therefore
$$\begin{aligned} \frac{10}{7} - \mathbf{x} \cdot \hat{\mathbf{n}} &= \frac{10 - ((-2) \times 2 + 4 \times 6 + 5 \times (-3))}{7} \\ &= \frac{5}{7}. \end{aligned}$$

2. If λ is an eigenvalue of an orthogonal matrix A with eigenvector $\mathbf{v}$, then
$$A\mathbf{v} = \lambda \mathbf{v}.$$
Taking the transpose of both sides gives
$$\mathbf{v}^T A^T = \lambda \mathbf{v}^T.$$
Applying this to the original equation gives
$$\mathbf{v}^T A^T A \mathbf{v} = \lambda^2 \mathbf{v}^T \mathbf{v}$$
and since $A^T A = I$ it follows that
$$|\mathbf{v}|^2 = \lambda^2 |\mathbf{v}^2|.$$
Since $|\mathbf{v}| > 0$, it follows that $\lambda = \pm 1$.

3. The squares of the matrices are
$$\begin{aligned} A^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ B^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} (-i)i & 0 \\ 0 & i(-i) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ C^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1^2 & 0 \\ 0 & (-1)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$
and the products are
$$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad BC = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad CA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad CB = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad AC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It can be seen that $AB = -BA$, $BC = -CB$, and $CA = -AC$. In addition

$$\begin{aligned} AB - BA &= \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2iC \\ BC - CB &= \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2iA \\ CA - AC &= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2iB. \end{aligned}$$

4. Since $A^2 = I$, it follows that $A^{2n} = I$ and $A^{2n+1} = A$ for any integer n. Hence, by using the Taylor series expansions for sine and cosine,

$$\begin{aligned} \sin kA &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} A^{2n+1} k^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} A k^{2n+1}}{(2n+1)!} \\ &= A \sum_{n=0}^{\infty} \frac{(-1)^{n+1} k^{2n+1}}{(2n+1)!} \\ &= A \sin k \end{aligned}$$

and

$$\begin{aligned} \cos kA &= \sum_{n=0}^{\infty} \frac{(-1)^{n} A^{2n} k^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n} k^{2n}}{(2n)!} \\ &= I \sum_{n=0}^{\infty} \frac{(-1)^{n} k^{2n}}{(2n)!} \\ &= I \cos k. \end{aligned}$$

Since $e^{ikA} = \cos kA + i \sin kA$ it follows that

$$e^{ikA} = \begin{pmatrix} \cos k & i \sin k \\ i \sin k & \cos k \end{pmatrix}.$$

The exponential is given by

$$
\begin{aligned}
e^{kA} &= \sum_{n=0}^{\infty} \frac{k^n A^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{k^{2n} A^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{k^{2n+1} A^{2n+1}}{(2n+1)!} \\
&= I \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} + A \sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!} \\
&= I \cosh k + A \sinh k \\
&= \begin{pmatrix} \cosh k & \sinh k \\ \sinh k & \cosh k \end{pmatrix}
\end{aligned}
$$

5. The eigenvalues of M are given by

$$
0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 1 - \lambda^3
$$

and hence $\lambda = 1, \alpha, \alpha^2$ where $\alpha = e^{2\pi i/3}$. To find the eigenvector $\mathbf{v} = (u, v, w)$ corresponding to 1, consider

$$
\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}
$$

Row reduction of the matrix gives

$$
\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}
$$

and thus if $w = 1$, then $u = 1$ and $v = 1$ so $\mathbf{v} = (1, 1, 1)$ is an eigenvector. For the eigenvalue α, the corresponding matrix can be row reduced as

$$
\begin{pmatrix} -\alpha & 0 & 1 \\ 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\alpha^2 \\ 1 & -\alpha & 0 \\ 0 & 1 & -\alpha \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\alpha^2 \\ 0 & -\alpha & -\alpha^2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\alpha^2 \\ 0 & -\alpha & -\alpha^2 \\ 0 & 0 & 0 \end{pmatrix}
$$

and thus $\mathbf{v} = (1, \alpha^2, \alpha)$ is an eigenvector, by making use of the fact that $\alpha^3 = 1$. Similar considerations show that $(1, \alpha, \alpha^2)$ is an eigenvector for α^2.

6. (a) The positions of the masses obey the equations

$$
\begin{aligned}
m\ddot{x} &= -k_1 x + k_2(y - x) \\
m\ddot{y} &= -k_2(y - x) - k_3 y,
\end{aligned}
$$

which can be written in matrix form as

$$m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) For $m = 1$, the eigenvalues satisfy

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -(k_1+k_2+\lambda) & k_2 \\ k_2 & -(k_2+k_3+\lambda) \end{vmatrix} \\ &= (k_1+k_2+\lambda)(k_2+k_3+\lambda) - k_2^2 \\ &= \lambda^2 + (k_1+2k_2+k_3)\lambda + (k_1+k_2)(k_2+k_3) - k_2^2 \\ &= \lambda^2 + (k_1+2k_2+k_3)\lambda + k_1k_2 + k_2k_3 + k_3k_1 \end{aligned}$$

and hence

$$\begin{aligned} \lambda &= \frac{-(k_1+2k_2+k_3) \pm \sqrt{(k_1+2k_2+k_3)^2 - 4(k_1k_2+k_2k_3+k_3k_1)}}{2} \\ &= \frac{-(k_1+2k_2+k_3) \pm \sqrt{(k_1-k_3)^2 + 4k_2^2}}{2}. \end{aligned}$$

For $k_1 = 1$ and $k_2 = 2$ this becomes

$$\lambda = \frac{-3 - 2k_2 \pm \sqrt{1+4k_2^2}}{2}.$$

(c) The eigenvalues are shown in Fig. 1. For k_2, the eigenvalues are -1 and -2, corresponding to the natural vibrational frequencies of the two masses in the absence of any connection between them. As k_2 increases, it can be seen that one eigenvalue tends to -1.5. The connecting spring between the two masses becomes increasingly like a rigid rod, and thus the masses will oscillate in unison with each other. Together, they have a mass of $2m$, and are subjected to springs with a total spring constant of $k_1 + k_3 = 3$, and thus the frequency of oscillation is given by $3/2$. The other eigenvalue continues to increase according to $-2k_2$ and corresponds to a rapid vibrational mode within the central spring itself.

7. (a) The solutions of the equation $dy/dt = -y$ are given by

$$\frac{dy}{y} = -dt, \qquad \log y = -t + C, \qquad y(t) = ae^{-t}$$

where C and a are arbitrary constants. Now consider the set of these solutions, defining $y_a(t) = ae^{-t}$. It can be seen that adding two solutions results in another,

$$y_a(t) + y_b(t) = (a+b)e^{-t} = y_{a+b}(t),$$

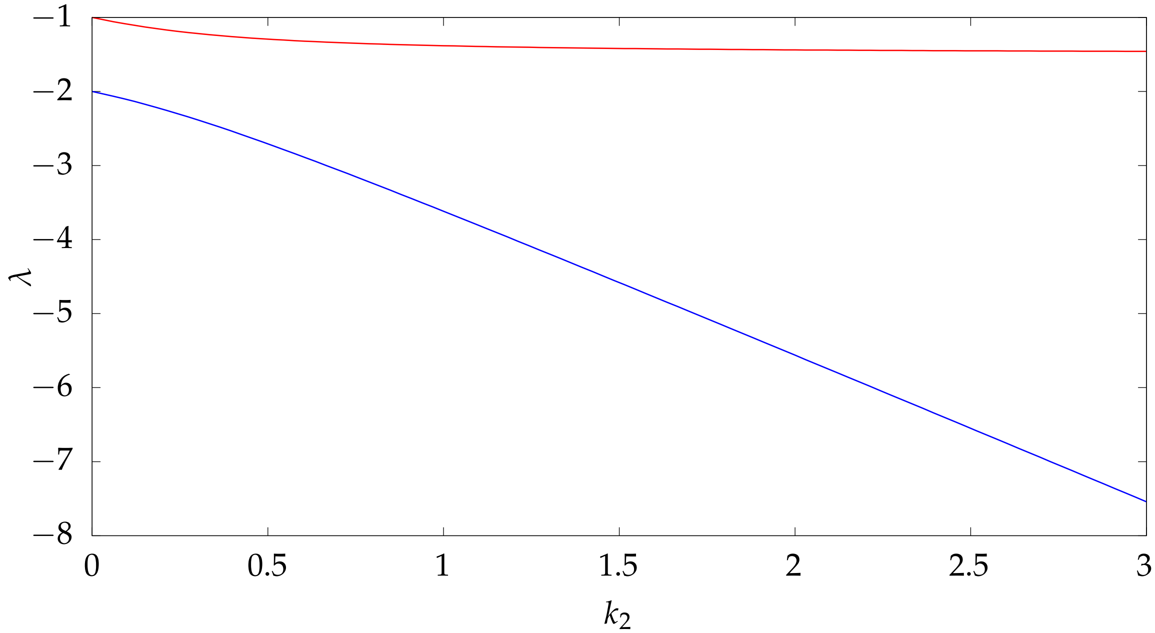

Figure 1: Eigenvalues for the coupled spring problem as a function of the central spring constant k_2.

and multiplying a solution by a scalar λ also results in another,

$$\lambda y_a(t) = \lambda a e^{-t} = y_{\lambda a}(t).$$

Since the solutions can be added and multiplied following the usual rules of arithmetic for real numbers, and the real numbers themselves form a vector space, the solutions must form a vector space.

(b) Due to the presence of the constant term, the differential equation $dy/dt = 1 - y$ is not linear, and thus it should not be expected that its solutions form a vector space. To see this, note that $y(t) = 1$ is a solution of $dy/dt = 1 - y$. If the T was a vector space, then $y(t) = 2$ would also be a solution, but

$$dy/dt = 0, \qquad 1 - y = 1.$$

Hence T does not form a vector space.

8. (a) Let the elements of A and B be a_{ij} and b_{ij} respectively. Then, using the summation convention,

$$[AB]_{ik} = a_{ij}b_{jk}, \qquad [BA]_{ik} = b_{ij}a_{jk}$$

and hence

$$\text{Tr}(AB) = a_{ij}b_{ji} = b_{ij}a_{ij} = \text{Tr}(BA).$$

(b) To show that it is not always the case that $\text{Tr}\, ABC = \text{Tr}\, CBA$, consider the Pauli spin matrices from Exercise 3. Using the previously established multiplicative properties, it can be seen that

$$\begin{aligned} \text{Tr}(ABC) &= \text{Tr}(iCC) = \text{Tr}(iI) = 2i \\ \text{Tr}(CBA) &= \text{Tr}(-iCC) = \text{Tr}(-iI) = -2i \end{aligned}$$

and thus the two expressions are not equal.

Now suppose that the matrix C has elements c_{ij}. By extending the argument from part (a), it can be seen that

$$[ABC]_{il} = a_{ij}b_{jk}c_{kl}, \qquad [CAB]_{il} = c_{ij}a_{jk}b_{kl}$$

and hence

$$\text{Tr}(ABC) = a_{ij}b_{jk}c_{kl} = c_{ij}a_{jk}b_{kl} = \text{Tr}(CAB).$$

(c) The trace only involves diagonal entries, so $\text{Tr}\, M = \text{Tr}\, M^T$. If S is symmetric and A is antisymmetric, then by making use of the result from part (a), it can be seen that

$$\text{Tr}(SA) = \text{Tr}((SA)^T) = \text{Tr}(A^T S^T) = \text{Tr}(-AS) = -\text{Tr}(AS) = -\text{Tr}(SA)$$

and hence $\text{Tr}(SA) = 0$.

9. Using the result from the previous exercise,

$$\mathrm{Tr}(C^{-1}MC) = \mathrm{Tr}(MCC^{-1}) = \mathrm{Tr}\, M.$$

Assume now that the matrix M can be diagonalized by C, so that $C^{-1}MC = D$. The diagonal matrix has the same eigenvalues as M, and they are located on the diagonal, so the sum of eigenvalues is $\mathrm{Tr}\, D$. By the above relation, it follows that $\mathrm{Tr}\, M = \mathrm{Tr}\, D$ and is the sum of eigenvalues of M.