Math 121A: Homework 3 solutions

1. The logarithm of i is
$$\log i = \frac{i\pi}{2} + 2\pi i n$$
where n is any integer. Hence
$$\sqrt[3]{i} = e^{(\log i)/3} = e^{i\pi/6 + 2\pi i n/3}$$
and by considering $n = 0, 1, 2$ it can be seen that
$$\sqrt[3]{i} = e^{i\pi/6}, e^{5i\pi/6}, e^{-i\pi/2}.$$
For other values of n, these solutions begin to repeat, due to the fact that $e^{2i\pi} = 1$. Similarly
$$\log(-i) = -\frac{i\pi}{2} + 2\pi i n$$
where n is any integer, and thus
$$\sqrt[3]{-i} = e^{-i\pi/6}, e^{i\pi/2}, e^{-5i\pi/6}.$$
The positions of the roots in the complex plane are shown in Fig. 1.

2. (a) Substituting $z = x + yi$ into $|z+1| = |z+i|$ gives
$$|x + yi + 1| = |x + yi + i|$$
and hence
$$(x+1)^2 + y^2 = x^2 + (y+1)^2.$$
Therefore
$$x^2 + 2x + 1 + y^2 = x^2 + y^2 + 2y + 1$$
and thus $x = y$. The locus of points satisfying this equation is a straight line, as shown in Fig. 2(a).

   (b) Substituting $z = x + yi$ into $\operatorname{Re} z = |z - 1|$ gives
$$x = |x - 1 + yi|$$
and hence
$$x^2 = (x-1)^2 + y^2.$$
Therefore $y^2 + 1 = 2x$. The locus of points satisfying this equation is a parabola, as shown in Fig. 2(b).

3. The linear system can be written as the augmented matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 2 & -2 & 4 & 6 \end{pmatrix},$$

which can then be row reduced according to

$$A \to \begin{pmatrix} -1 & 1 & -1 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the solutions are characterized by $z = 7$ and $y = 11 - x$. Thus there is an infinite one-dimensional family of solutions $(x, y, z) = (\lambda, 11 - \lambda, 7)$.

4. The matrix can be row reduced as follows:

$$\begin{pmatrix} 1 & 1 & 4 & 3 \\ 3 & 1 & 10 & 7 \\ 4 & 2 & 14 & 10 \\ 2 & 0 & 6 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 4 & 3 \\ 0 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 \\ 0 & -2 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 4 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence the rank of the matrix is 2.

5. The determinant can be evaluated as

$$\begin{aligned} \begin{vmatrix} 5 & 17 & 3 \\ 2 & 4 & -3 \\ 11 & 0 & 2 \end{vmatrix} &= 5 \times 4 \times 2 + 17 \times (-3) \times 11 + 3 \times 2 \times 0 \\ &\quad -5 \times (-3) \times 0 - 17 \times 2 \times 2 - 3 \times 4 \times 11 \\ &= -721. \end{aligned}$$

6. First, note that

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= e^{3i\theta} \\ &= (e^{i\theta})^3 \\ &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta - 3i\sin^3\theta \end{aligned}$$

and equating the real parts implies that

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta \\ &= \cos^3\theta - 3(1 - \cos^2\theta)\cos\theta \\ &= 4\cos^3\theta - 3\cos\theta. \end{aligned}$$

By making use of this identity, the given determinant can be evaluated as

$$\begin{aligned} \begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix} &= 4\cos^3\theta + 0 + 0 - 2\cos\theta - \cos\theta \\ &= 4\cos^3\theta - 3\cos\theta \\ &= \cos 3\theta. \end{aligned}$$

7. (a) Since the combined age of Alice, Bob, and Charlie is 18, it follows that

$$a + b + c = 18. \tag{1}$$

Since Charlie is older than Bob, the positive difference between their ages is $c - b$, and thus if Alice's age is twice this, then

$$a = 2(c - b). \tag{2}$$

If Bob's age is the average of Alice's and Charlie's ages, then

$$b = \frac{a + c}{2}. \tag{3}$$

These three equations can be written as the linear system

$$\begin{aligned} a + b + c &= 18 \\ a + 2b - 2c &= 0 \\ a - 2b + c &= 0. \end{aligned}$$

(b) The linear system can be expressed as the augmented matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 18 \\ 1 & 2 & -2 & 0 \\ 1 & -2 & 1 & 0 \end{pmatrix},$$

which can be row reduced according to

$$\begin{aligned} A &\to \begin{pmatrix} 1 & 1 & 1 & 18 \\ 0 & 1 & -3 & -18 \\ 0 & -3 & 0 & -18 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 18 \\ 0 & 1 & -3 & -18 \\ 0 & 0 & -9 & -72 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 1 & 1 & 18 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \end{aligned}$$

and thus Alice is four, Bob is six, and Charlie is eight.

8. The vectors are

$$-\mathbf{A} = -2\mathbf{i} - 3\mathbf{j}, \qquad 3\mathbf{B} = 12\mathbf{i} - 12\mathbf{j}, \qquad \mathbf{A} - \mathbf{B} = -2\mathbf{i} + 7\mathbf{j},$$

$$\mathbf{B} + 2\mathbf{A} = 8\mathbf{i} + 2\mathbf{j}, \qquad \frac{\mathbf{A} + \mathbf{B}}{2} = 3\mathbf{i} + \frac{1}{2}\mathbf{j}.$$

They are plotted in Fig. 3.

9. Note that

$$\begin{aligned}(\mathbf{B}|\mathbf{A}| + \mathbf{A}|\mathbf{B}|) \cdot (\mathbf{A}|\mathbf{B}| - \mathbf{B}|\mathbf{A}|) &= \mathbf{B} \cdot \mathbf{B}|\mathbf{A}|^2 + \mathbf{A} \cdot \mathbf{B}|\mathbf{A}||\mathbf{B}| \\ &\quad -\mathbf{A} \cdot \mathbf{B}|\mathbf{A}||\mathbf{B}| - \mathbf{A} \cdot \mathbf{A}|\mathbf{B}|^2 \\ &= \mathbf{B} \cdot \mathbf{B}|\mathbf{A}|^2 - \mathbf{A} \cdot \mathbf{A}|\mathbf{B}|^2 \\ &= 0.\end{aligned}$$

Since the scalar product of these two vectors is zero, it follows that they must be orthogonal.

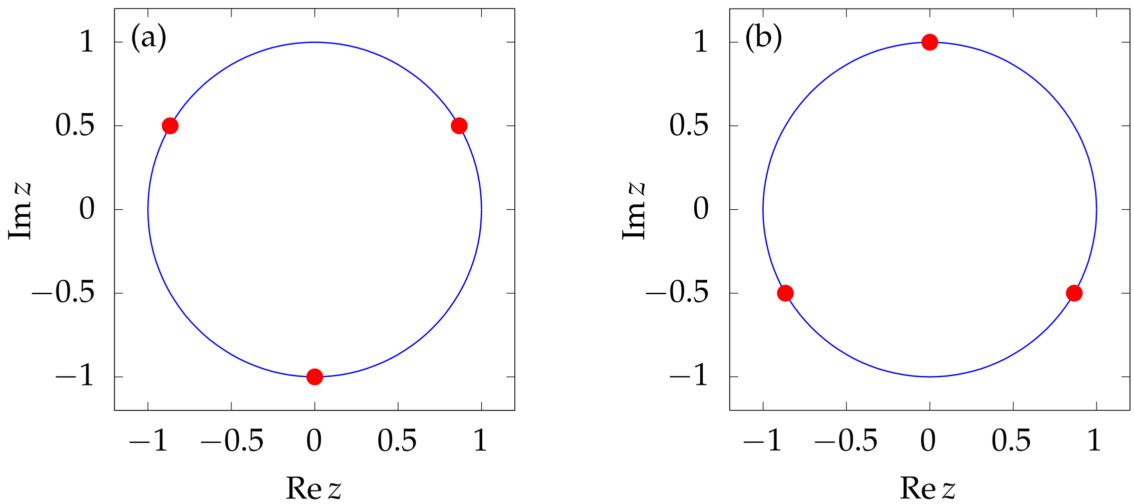

Figure 1: Positions of (a) the roots $\sqrt[3]{i}$ and (b) the roots of $\sqrt[3]{-i}$ in the complex plane. The blue circle is the locus of $|z| = 1$, showing that all of the solutions lie on this.

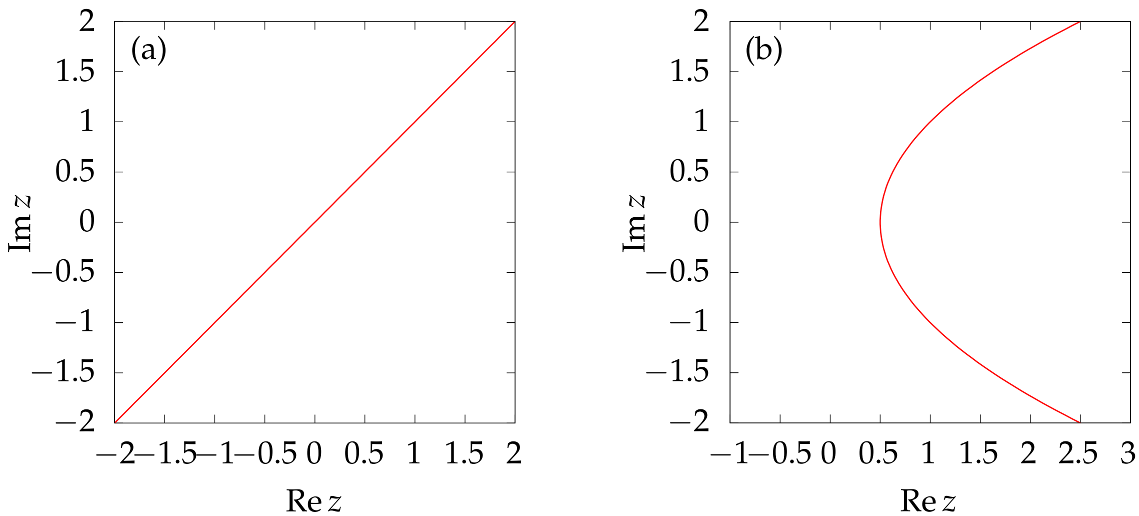

Figure 2: Loci of (a) $|z + 1| = |z + i|$ and (b) $\operatorname{Re} z = |z - 1|$.

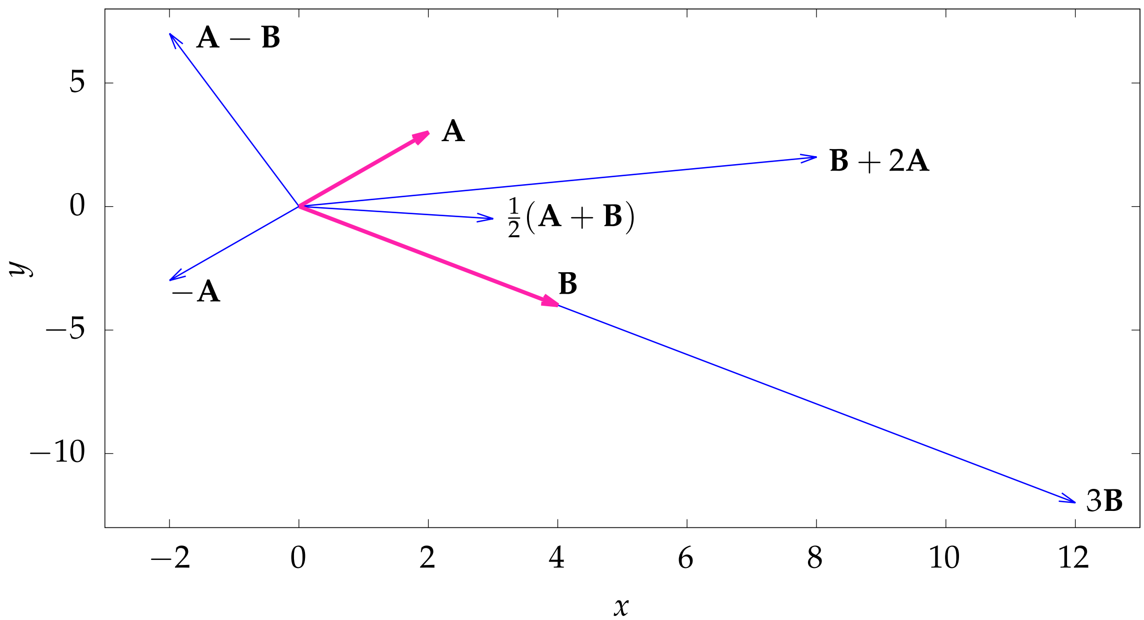

Figure 3: Plots of the vectors $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} - 4\mathbf{j}$, plus a number of linear combinations.