Math 121A: Homework 2 solutions

1. (a) By considering terms up to $O(x^3)$, the Taylor series is

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right) + O(x^4) \\ &= x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} + O(x^4) \\ &= x + x^2 + \frac{x^3}{3} + O(x^4). \end{aligned}$$

A comparison between the function and the Taylor series is shown in Fig. 1(a).

(b) First put $y = x + x^2$. Then, keeping terms up to $O(x^3)$, the Taylor series is

$$\begin{aligned} \frac{1}{1+x+x^2} &= \frac{1}{1+y} \\ &= 1 - y + y^2 - y^3 + O(y^4) \\ &= 1 - (x+x^2) + (x+x^2)^2 - (x+x^2)^3 + O(x^4) \\ &= 1 - x - x^2 + x^2 + 2x^3 - x^3 + O(x^4) \\ &= 1 - x + x^3 + O(x^4). \end{aligned}$$

A comparison between the function and the Taylor series is shown in Fig. 1(b).

(c) The Taylor series is given by

$$\begin{aligned} \sin(\log(1+x)) &= \sin\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + O(x^4) \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) - \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)^3}{6} + O(x^4) \\ &= x - \frac{x^2}{2} + \frac{x^3}{6} + O(x^4). \end{aligned}$$

A comparison between the function and the Taylor series is shown in Fig. 1(c).

2. For the Taylor's series up to the term in x^k, the remainder can be written as

$$R_k(x) = \frac{f^{(k+1)}(c)x^{k+1}}{(k+1)!}.$$

Since the derivatives of $f(x) = \sin x$ are all either $\pm\sin x$ or $\pm\cos x$, it follows that $|f^{(k+1)}(c)| \le 1$, and hence

$$|R_k(x)| \le \frac{|x|^{k+1}}{(k+1)!}.$$

For a fixed value of x, this will converge to zero as k increases, since eventually the factorial will dominate the $|x|^{k+1}$ term.

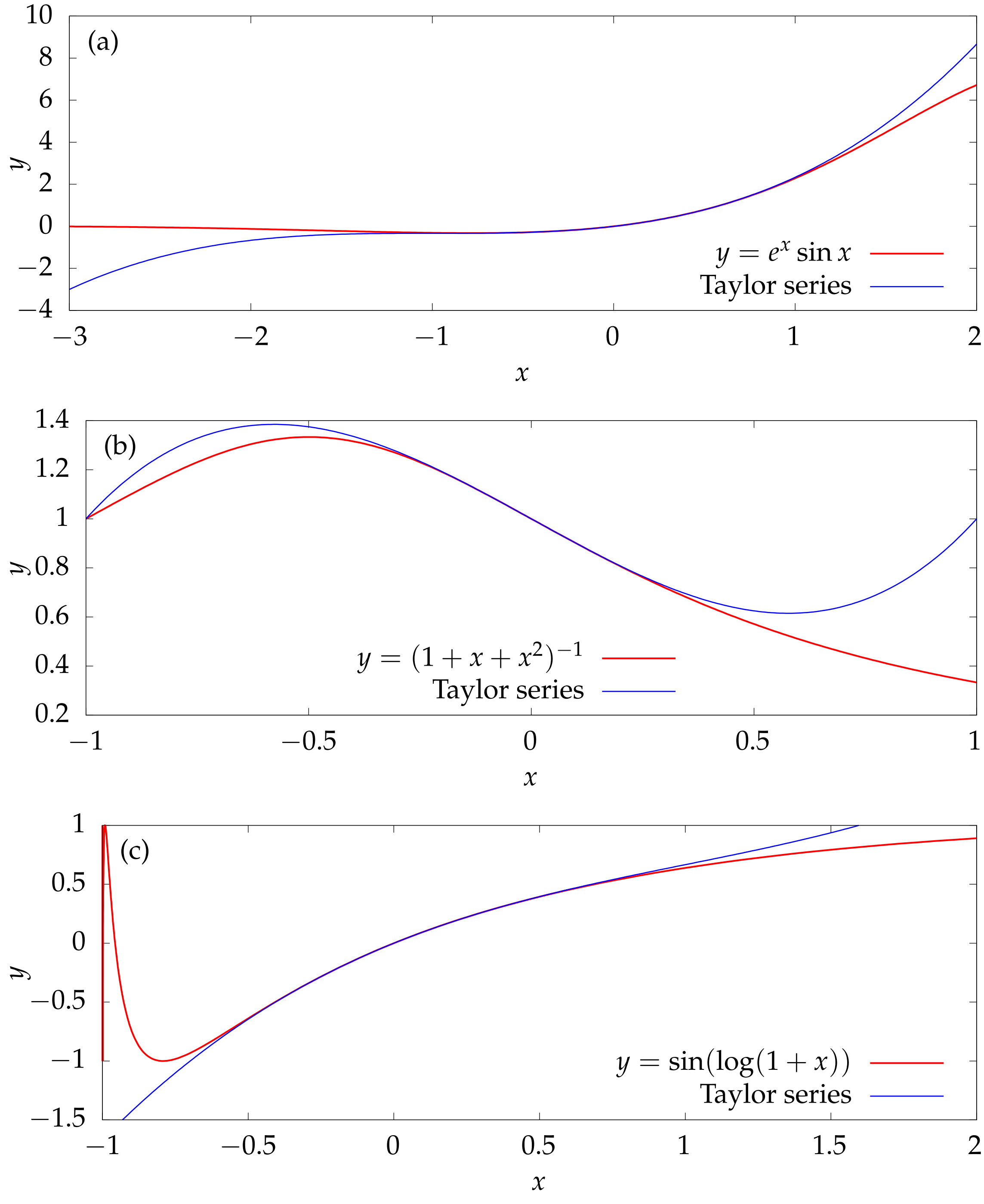

Figure 1: Third order Taylor series approximations for the functions considered in question 1. For each case, the Taylor series closely matches the function in the region near $x = 0$.

3. By reference to the diagram, the angle θ satisfies

$$\cos\theta = \frac{R}{R+h},$$

and since θ will be small for a physical distance, it can be approximated using the Taylor series as

$$1 - \frac{\theta^2}{2} = \frac{R}{R+h},$$

which can be rearranged to give

$$\theta^2 = 2 - \frac{2R}{R+h} = \frac{2R+2h-2R}{R+h} = \frac{2h}{R+h}.$$

Since $h \ll R$, then $R+h$ can be replaced by R, and the angle can be approximated as

$$\theta = \sqrt{\frac{2h}{R}}.$$

Hence the distance that can be seen along the surface of the Earth is

$$s = R\theta = \sqrt{2hR}.$$

The Earth's radius is $3,959$ mi, and there are 5,280 feet in a mile. Hence, if h is measured in feet, then the distance in miles that can be seen is

$$s = \sqrt{2 \times \frac{h}{5,280} \times 3,959} \approx \sqrt{1.4996h} \approx \sqrt{\frac{3h}{2}}.$$

4. (a) If $x = 1$, then the given Taylor series can by evaluated as

$$\frac{\pi}{4} = \tan^{-1} 1 \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} = 0.7542679\ldots$$

and hence an approximation for π is

$$\pi \approx 3.0170718\ldots.$$

This is a very poor approximation, since the terms in the expansion only decay very slowly.

(b) For a million terms, the series gives the approximation

$$\pi \approx 3.14159165358977\ldots$$

which agrees with π up to the first five decimal places.

(c) For the given complex number,

$$(3+i)^2(7+i) = (8+6i)(7+i) = 50+50i.$$

Since the real and imaginary parts of this number are the same, $\text{Arg}(50+50i) = \pi/4$. Since arguments of complex numbers add together when they are multiplied, it follows that

$$\frac{\pi}{4} = \text{Arg}\left((3+i)^2(7+i)\right) = 2\,\text{Arg}(3+i) + \text{Arg}(7+i) = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}.$$

(d) Evaluating the first four terms in each series, it can be found that

$$\begin{aligned} \frac{\pi}{4} &\approx 2\left(1/3 - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7}\right) \\ &\quad + 1/7 - \frac{(1/7)^3}{3} + \frac{(1/7)^5}{5} - \frac{(1/7)^7}{7} \\ &\approx 0.785387808966927\ldots \end{aligned}$$

and hence

$$\pi \approx 3.14155123586771\ldots.$$

This is a much better approximation than in (a), and matches π to the first four decimal places—it is almost as accurate as the million-term expansion in (b). The presence of the factors of 1/3 and 1/7 mean that the terms decay much more quickly.

5. The complex numbers z and w are related by

$$w = \frac{1+iz}{i+z}.$$

Write $z = x+iy$ and $w = u+iv$ where u, v, x, and y are real.

(a) Substituting $z = x+iy$ gives

$$\begin{aligned} u+iv &= \frac{1+i(x+iy)}{i+x+iy} \\ &= \frac{1-y+ix}{x+i(y+1)} \\ &= \frac{(x-i(y+1))(1-y+ix)}{x^2+(y+1)^2} \\ &= \frac{2x+i(x^2+y^2-1)}{x^2+(y+1)^2} \end{aligned}$$

and hence

$$u = \frac{2x}{x^2+(y+1)^2}, \qquad v = \frac{x^2+y^2-1}{x^2+(y+1)^2}.$$

(b) Substituting $x = \tan(\theta/2)$ and $y = 0$ gives

$$u = \frac{2\tan(\theta/2)}{\tan^2(\theta/2)+1} = \frac{2\tan(\theta/2)}{\sec^2(\theta/2)} = 2\sin(\theta/2)\cos(\theta/2) = \sin\theta,$$

where several trigonometric identities have been employed. Similarly

$$v = \frac{\tan^2(\theta/2)-1}{\tan^2(\theta/2)+1} = \frac{\tan^2(\theta/2)-1}{\sec^2(\theta/2)} = \sin^2(\theta/2) - \cos^2(\theta/2) = -\cos\theta.$$

The real axis is parameterized by values of θ in the range $-\pi < \theta < \pi$. This will trace out the circle given by $(u, v) = (\sin\theta, -\cos\theta)$ with the exception of the point when $\theta = \pi$, corresponding to $(u, v) = (0, 1)$, which is $w = i$.