Math 121A: Homework 11 solutions

1. For $F(x, y, y') = y'^2 + y^2$ the Euler Lagrange equation is

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}(2y') - 2y = 2(y'' - y).$$

This is a second-order linear differential equation. Substituting $y = e^{mx}$, it can be seen that $m^2 = 1$ and thus $m = \pm 1$. The general solution is therefore

$$y(x) = Ae^x + Be^{-x}$$

for some constants A and B.

2. (a) If $r' = dr/d\theta$, the given functional can be written as

$$F(\theta, r, r') = \sqrt{r^2 + r'^2}$$

and hence the Euler equation gives

$$\begin{aligned} 0 &= \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) - \frac{\partial F}{\partial r} \\ &= \frac{d}{d\theta}\left(\frac{r'}{\sqrt{r^2 + r'^2}}\right) - \frac{r}{\sqrt{r^2 + r'^2}} \\ &= \frac{r''}{\sqrt{r^2 + r'^2}} - \frac{r'(r + r')}{(r^2 + r'^2)^{3/2}} - \frac{r}{\sqrt{r^2 + r'^2}}. \end{aligned}$$

Multiplying through by $(r^2 + r'^2)^{3/2}$ gives

$$\begin{aligned} 0 &= r''(r^2 + r'^2) - r'(rr' + r'r'') - r(r^2 + r'^2) \\ &= r''r^2 + r''r'^2 - rr'^2 - r''r'^2 - r^3 - rr'^2 \\ &= r''r^2 - 2rr'^2 - r^3 \end{aligned}$$

and by removing the common factor of r, the equation

$$r''r - 2r'^2 - r^2 = 0 \qquad (1)$$

is obtained. An alternative equation can be obtained by using the Beltrami identity,

$$C = F - r'\frac{\partial F}{\partial r'} = \sqrt{r^2 + r'^2} - \frac{r'^2}{\sqrt{r^2 + r'^2}} = \frac{r^2}{\sqrt{r^2 + r'^2}} \qquad (2)$$

for some constant C.

(b) The straight line from $(1,-1)$ to $(1,1)$ can be expressed in polar coordinates as

$$r(\theta) = \frac{1}{\cos\theta} = \sec\theta$$

for the range $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$. The first and second derivatives are given by

$$r'(\theta) = \frac{\sin\theta}{\cos^2\theta}, \qquad r''(\theta) = \frac{1}{\cos\theta} + \frac{2\sin^2\theta}{\cos^3\theta}.$$

Substituting into Eq. 1 gives

$$r''r - 2r'^2 - r^2 = \frac{1}{\cos^2\theta} + \frac{2\sin^2\theta}{\cos^4\theta} - 2\frac{\sin^2\theta}{\cos^4\theta} - \frac{1}{\cos^2\theta} = 0$$

and the equation is satisfied. Substituting into Eq. 2 gives

$$\frac{r^2}{\sqrt{r^2 + r'^2}} = \frac{\sec^2\theta}{\sqrt{\frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^4\theta}}} = \frac{1}{\sqrt{\cos^2\theta + \sin^2\theta}} = 1$$

and the equation is also satisfied.

3. (a) For $F(x,y,y') = g(x)\sqrt{1+y'^2}$, the Euler equation gives

$$0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{d}{dx}\left(\frac{g(x)y'}{\sqrt{1+y'^2}}\right)$$

and hence

$$\frac{g(x)y'}{\sqrt{1+y'^2}} = C$$

for some constant C. Therefore

$$[g(x)]^2 y'^2 = C^2 + y'^2 C^2$$

and

$$y'^2 = \frac{C^2}{[g(x)]^2 - C^2}.$$

Hence

$$y' = \frac{C}{\sqrt{[g(x)]^2 - C^2}}.$$

(b) For $g(x) = 1$, the equation becomes

$$y' = \frac{C}{\sqrt{1 - C^2}}$$

which is equivalent to $y' = D$ for some constant D, and hence $y(x) = Dx + E$ for some constant E. This will satisfy the boundary conditions for

$$y(x) = 2x - 2.$$

If $g(x) = \sqrt{x}$, the equation is

$$y' = \frac{C}{\sqrt{x - C^2}}$$

and hence

$$y(x) = \int \frac{C}{\sqrt{x - C^2}}\, dx = 2C\sqrt{x - C^2} + D.$$

The boundary conditions give

$$0 = y(1) = 2C\sqrt{1 - C^2} + D, \qquad 2 = y(2) = 2C\sqrt{2 - C^2} + D$$

If the hint is used, and D is assumed to be zero, then it can be seen that $C = 1$ is the only solution, and thus

$$y(x) = 2\sqrt{x - 1}.$$

Without the hint, the solution can be found by eliminating D to obtain

$$2C\sqrt{2 - C^2} = 2 + 2C\sqrt{1 - C^2}. \tag{3}$$

Dividing by two and squaring both sides gives

$$C^2(2 - C^2) = 1 + C^2(1 - C^2) + 2C\sqrt{1 - C^2}$$

and hence

$$C^2 - 1 = 2C\sqrt{1 - C^2}.$$

Squaring both sides again gives

$$C^4 - 2C^2 + 1 = 4C^2(1 - C^2),$$

which simplifies to

$$5C^4 - 6C^2 + 1 = 0.$$

This can be factorized as

$$(5C^2 - 1)(C^2 - 1) = 0$$

and thus $C = \pm 1, \pm\sqrt{1/5}$. Since squaring both sides may introduce more solutions, these possible values for C can then be checked that they indeed satisfy Eq. 3. It can be verified that $C = 1$ is the only valid solution.

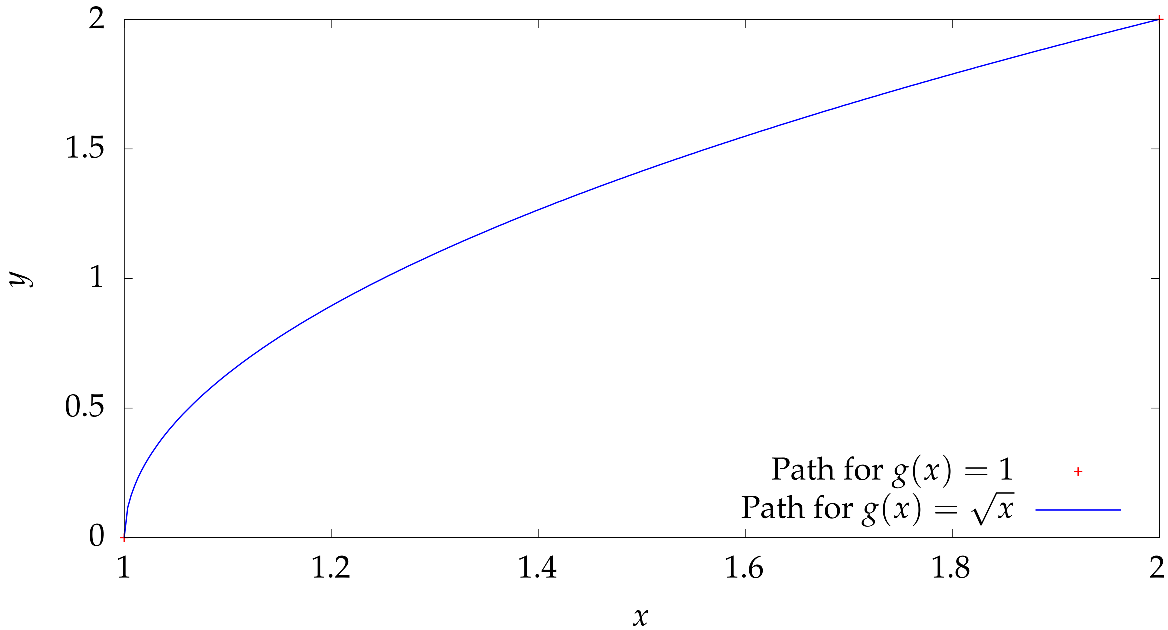

Figure 1: Extremal paths for the weighted geodesics considered in question 3.

(c) The two solutions are plotted in Fig. 1. It can be seen that the path for $g(x) = \sqrt{x}$ increases more quickly near $x = 1$ before leveling off near $x = 2$. This should be expected since $\sqrt{x}$ is smaller near $x = 1$, and thus y is able to grow more quickly in this region without incurring a large penalty in the integral.

4. (a) To derive the more general Euler equation, assume first that $y(x)$ is an extremal path, and consider $Y(x) = y(x) + \epsilon\eta(x)$ where ϵ is a small parameter and $\eta(x)$ is a variation that is twice-differentiable. Assume further that $\eta(x_1) = \eta(x_2) = 0$ and that $\eta'(x_1) = \eta'(x_2) = 0$. Then

$$I[Y] = \int_{x_1}^{x_2} F(x, Y, Y', Y'')dx = \int_{x_1}^{x_2} F(x, y + \epsilon\eta, y' + \epsilon\eta', y'' + \epsilon\eta'')dx$$

and

$$0 = \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} = \int_{x_1}^{x_2} \left(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'} + \eta''\frac{\partial F}{\partial y''}\right) dx. \tag{4}$$

Consider the second and third terms in this integral. Applying integration by parts to the second term gives

$$\begin{aligned}\int_{x_1}^{x_2} \eta'\frac{\partial F}{\partial y'}dx &= \left[\eta\frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx \\ &= -\int_{x_1}^{x_2} \eta\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx.\end{aligned}$$

Applying integration by parts twice to the third term gives

$$\begin{aligned}\int_{x_1}^{x_2} \eta'' \frac{\partial F}{\partial y''} dx &= \left[\eta' \frac{\partial F}{\partial y''}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta' \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right) dx \\ &= -\left[\eta \frac{d}{dx}\left(\frac{\partial F}{\partial y''}\right)\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \eta \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx \\ &= \int_{x_1}^{x_2} \eta \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) dx.\end{aligned}$$

Therefore Eq. 4 can be written as

$$0 = \int_{x_1}^{x_2} \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)\right) dx.$$

For this to be true for all possible variations $\eta(x)$, it follows that

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0.$$

(b) For the case of $F(x, y, y', y'') = \frac{(y'')^2}{2}$, the equation from part (a) can be used to obtain

$$\frac{d^2}{dx^2}(y'') = 0$$

and thus

$$\frac{d^4 y}{dx^4} = 0.$$

By integrating four times, it can be seen that the general solution is a cubic,

$$y(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

To satisfy the boundary conditions $y(0) = 1$ and $y'(0) = 0$ it can be seen that $a_1 = 0$ and $a_0 = 1$. The condition $y(1) = -1$ gives

$$-1 = a_3 + a_2 + 1$$

and the condition $y'(1) = 0$ gives

$$0 = 3a_3 + 2a_2,$$

and hence $a_3 = -4$ and $a_2 = -6$, so the solution is

$$y(x) = 1 - 6x^2 + 4x^3.$$

It can be seen that

$$y'(x) = -12x + 12x^2, \qquad y''(x) = -12 + 24x$$

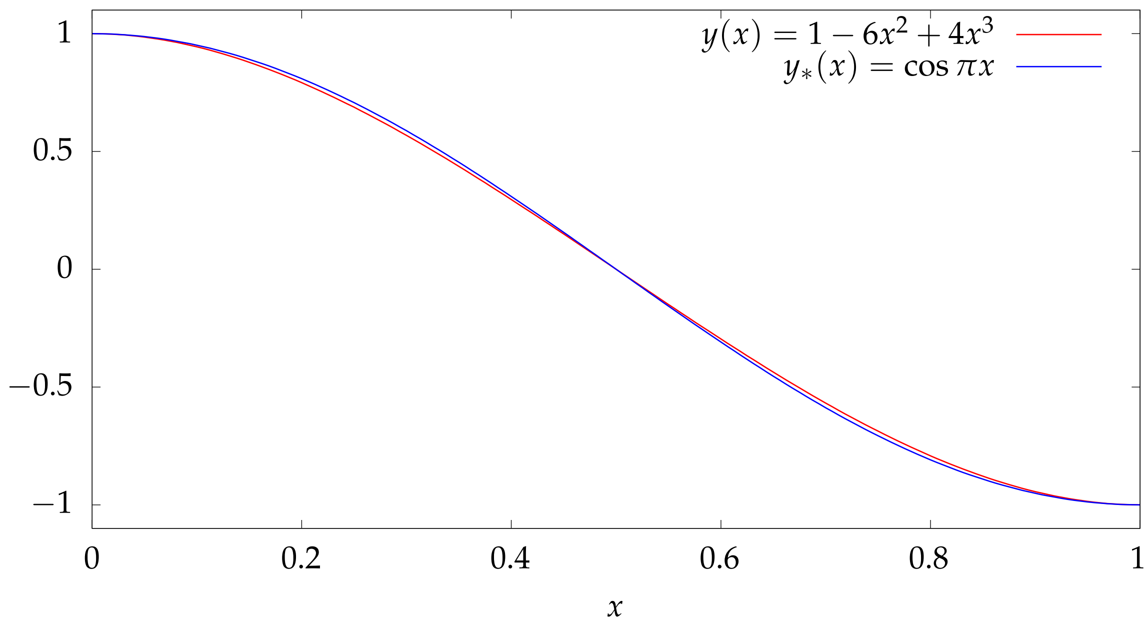

Figure 2: Graph of the two functions y and y_* considered in question 4.

and hence

$$
\begin{aligned}
I[y] &= \frac{1}{2}\int_0^1 (-12 + 24x)^2 dx \\
&= \frac{144}{2}\int_0^1 (1 - 4x + 4x^2) dx \\
&= 72\left(1 - 2 + \frac{4}{3}\right) \\
&= \frac{72}{3} = 24.
\end{aligned}
$$

(c) If $y_*(x) = \cos \pi x$ then $y_*'(x) = -\pi \sin \pi x$ and hence

$$
y_*(0) = \cos 0 = 1, \qquad y_*(0) \cos \pi = -1,
$$

$$
y_*'(0) = -\pi \sin 0 = 0, \qquad y_*'(0) = -\pi \sin \pi = 0
$$

so the function satisfies the given boundary conditions. It can be seen that

$y_*''(x) = -\pi^2 \cos \pi x$ and hence

$$\begin{aligned} I[y_*] &= \frac{1}{2}\int_0^1 (y_*'')^2 dx \\ &= \frac{1}{2}\int_0^1 \pi^4 \cos^2 \pi x \, dx \\ &= \frac{\pi^4}{2}\int_0^1 \frac{1}{2}\left(1+\cos 2\pi x\right) dx \\ &= \frac{\pi^4}{4}\left[x + \frac{\sin 2\pi x}{2\pi}\right]_0^1 \\ &= \frac{\pi^4}{4} \\ &= 24.35227\ldots. \end{aligned}$$

Figure 2 shows plots of y and y_*. It can be seen that the curves are similar, and thus it should be expected that $I[y_*]$ is close to $I[y]$ in value. It can also be seen that $I[y_*] > I[y]$ as would be expected since y is an extremal path, and in this case a minimal path.

5. Assume that the curve is between the two points $(x, y) = (\pm a, 0)$, and that its shape is given by $y(x)$, so that $y(\pm a) = 0$. The problem involves maximizing

$$I[y] = \int_{-a}^{a} 2\pi y \sqrt{1 + y'^2} dx$$

subject to the constraint

$$J[y] = \int_{-a}^{a} \sqrt{1 + y'^2} dx = l.$$

This can be carried out by introducing a Lagrange multiplier λ and considering

$$\begin{aligned} K[y, \lambda] &= I[y] + \lambda(J[y] - l) \\ &= -\lambda l + \int_{-a}^{a} (2\pi y + \lambda)\sqrt{1 + y'^2} dx \\ &= \int_{-a}^{a} F(\lambda, x, y, y') dx \end{aligned}$$

where $F(\lambda, x, y, y') = (2\pi y + \lambda)\sqrt{1 + y'^2}$. Since there is no explicit x dependence in F, the Beltrami identity can be used: if C is a constant then

$$\begin{aligned} C &= F - y' \frac{\partial F}{\partial y'} \\ &= (2\pi y + \lambda)\sqrt{1 + y'^2} - \frac{(2\pi y + \lambda) y'^2}{\sqrt{1 + y'^2}} \\ &= \frac{2\pi y + \lambda)}{\sqrt{1 + y'^2}}. \end{aligned}$$

Hence

$$C^2(1+y'^2) = (2\pi y + \lambda)^2$$

which can be rearranged to give

$$C\frac{dy}{dx} = \sqrt{(2\pi y + \lambda)^2 - C^2}.$$

Then

$$\int \frac{dx}{C} = \int \frac{dy}{\sqrt{(2\pi y + \lambda)^2 - C^2}}$$

and by making the substitution $2\pi y + \lambda = C\cosh u$, so that $2\pi dy = C\sinh u\, du$,

$$\frac{x - x_0}{C} = \frac{1}{2\pi}\int \frac{C\sinh u}{C\sinh u} du = \frac{1}{2\pi}\int du = \frac{u}{2\pi}$$

and hence

$$y(x) = \frac{-\lambda + C\cosh\frac{2\pi(x-x_0)}{C}}{2\pi}$$

where x_0 is a constant. From the conditions $y(\pm a) = 0$, it follows that $x_0 = 0$ and the solution is even. Furthermore

$$0 = y(a) = \frac{-\lambda + C\cosh\frac{2\pi a}{C}}{2\pi}$$

and hence

$$\lambda = C\cosh\frac{2\pi a}{C}.$$

To determine C, the constraint can be considered, which gives

$$\begin{aligned} l &= \int_{-a}^{a} \sqrt{1+y'^2}dx = \int_{-a}^{a} \sqrt{1+\sinh^2\frac{2\pi x}{C}}dx \\ &= \int_{-a}^{a} \cosh\frac{2\pi x}{C}dx = \frac{C}{2\pi}\left[\sinh\frac{2\pi x}{C}\right]_{-a}^{a} = \frac{C}{\pi}\sinh\frac{2\pi a}{C}. \end{aligned}$$

Writing $\alpha = 2\pi a/C$, this can written as

$$\frac{l}{2a} = \frac{\sinh\alpha}{\alpha}.$$

While it is not possible to write down an analytic expression for α there will be a solution to this equation for $l/2a \geq 1$, which should be expected, since the straight-line distance between the two endpoints is $2a$. Once α is determined, then an explicit value for C can be determined. Hence the solution is

$$y(x) = \frac{C}{2\pi}\left(\cosh\frac{2\pi x}{C} - \cosh\frac{2\pi a}{C}\right).$$

where C is a function of l and a.

6. (a) The kinetic energy and potential energy are given by

$$K = \frac{ma^2\dot{\theta}^2}{2}, \qquad V = -mga\cos\theta$$

respectively, where g is the gravitational acceleration. Hence the Lagrangian is

$$L(t,\theta,\dot{\theta}) = K - V = \frac{ma^2\dot{\theta}^2}{2} + mga\cos\theta.$$

The Euler–Lagrange equation therefore gives

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(ma^2\dot{\theta}\right) - (-mga\sin\theta) = ma^2\ddot{\theta} + mga\sin\theta$$

and hence

$$a\ddot{\theta} + g\sin\theta = 0.$$

(b) If θ is small then $\sin\theta \approx \theta$ and the equation becomes

$$a\ddot{\theta} + g\theta = 0.$$

This is a linear second-order differential equation with a solution

$$\theta(t) = A\cos\lambda t + B\sin\lambda t$$

where $\lambda = \sqrt{g/a}$. Hence the frequency of oscillations is given by

$$\frac{\lambda}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{a}}.$$

(c) For the case where L has no explicit t dependence, the Beltrami identity is

$$L - \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = C$$

for some constant C. In this case

$$\frac{ma^2\dot{\theta}^2}{2} + mga\cos\theta - \dot{\theta}\left(ma^2\dot{\theta}\right) = C$$

and hence

$$\frac{ma^2\dot{\theta}^2}{2} - mga\cos\theta = -C.$$

This is equivalent to

$$K + V = \text{const.}$$

expressing conservation of energy.