Math 121A: Homework 10 solutions

1. By using symmetry and making the substitution $z = e^{i\theta}$ the integral can be rewritten as

$$I = \int_0^{\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{1}{2i}\oint_C \frac{dz}{(1 - r(z + z^{-1}) + r^2)z}$$

where C is the unit circle. Hence

$$I = \frac{1}{2i}\oint_C \frac{dz}{-rz^2 + (1 + r^2)z - r} = \frac{1}{2i}\oint_C \frac{dz}{(rz - 1)(r - z)}.$$

The integrand has a simple pole at $z = r$. In addition, if $0 < r < 1$, the integral also has a simple pole at $z = 1/r$, but since $|1/r| > 1$ it lies outside the unit circle and its residue will not contribute. For $z = r$, the residue can be evaluated as

$$\text{Res}\left(\frac{1}{(rz - 1)(r - z)}, z = r\right) = \lim_{z \to r} \frac{z - r}{(rz - 1)(r - z)} = \frac{1}{1 - r^2}.$$

Hence, by the residue theorem,

$$I = \frac{1}{2i}\left(\frac{2\pi i}{1 - r^2}\right) = \frac{\pi}{1 - r^2}.$$

2. The integral can be written as

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} = \int_{-\infty}^{\infty} \frac{dx}{(x + 2 - i)(x + 2 + i)}$$

and thus the integrand has simple poles at $z = -2 + i$ and $z = -2 - i$. To use the residue theorem, consider the closing the integral with a large semicircle in the upper half plane; since the integrand decays according to $1/z^2$ as $|z| \to \infty$, the integral around this semicircle will vanish in the limit. Hence the original integral can be given in terms of the residues at poles in the upper half plane, namely

$$\begin{aligned} \text{Res}\left(\frac{1}{z^2 + 4z + 5}, z = -2 + i\right) &= \lim_{z \to -2+i} \frac{z + 2 - i}{(z + 2 - i)(z + 2 + i)} \\ &= \frac{1}{-2 + i + 2 + i} \\ &= \frac{1}{2i}. \end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 4x + 5} = 2\pi i\left(\frac{1}{2i}\right) = \pi.$$

3. By using the substitution $z = y/\sqrt{2}$, the Fourier integral can be written as

$$\tilde{f}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha x}dx}{x^4+1} = \frac{1}{2\sqrt{2}\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha y/\sqrt{2}}dy}{\frac{y^4}{4}+1} = \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\beta y}dy}{y^4+4}$$

where $\beta = \alpha/\sqrt{2}$. The integrand has simple poles at $y^4 = -4$, which corresponds to $(y^2-2i)(y^2+2i) = 0$, and $y = 1+i, 1-i, -1+i, -1-i$. Therefore

$$\tilde{f}(\alpha) = \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-i\beta y}dy}{(y-1-i)(y-1+i)(y+1-i)(y+1+i)}.$$

For $\alpha > 0$, the exponential will become small in the lower half plane, and the integral can be closed with a large semicircle in the lower half plane. This will enclose the simple poles at $y = 1-i$ and $y = -1-i$. The residues at these two points are

$$\begin{aligned}
\text{Res}\left(\frac{e^{-i\beta y}}{y^4+4}, y = 1-i\right) &= \lim_{x\to 1-i}\frac{e^{-i\beta y}}{(y-1-i)(y+1-i)(y+1+i)} \\
&= \frac{e^{-i\beta(1-i)}}{2(2-2i)(-2i)} \\
&= \frac{e^{\beta(-1-i)}}{-8-8i} = \frac{e^{\beta(-1-i)}(-1+i)}{16}
\end{aligned}$$

and

$$\begin{aligned}
\text{Res}\left(\frac{e^{-i\beta y}}{y^4+4}, y = -1-i\right) &= \lim_{x\to -1-i}\frac{e^{-i\beta y}}{(y-1-i)(y-1+i)(y+1-i)} \\
&= \frac{e^{-i\beta(-1-i)}}{(-2)(-2-2i)(-2i)} \\
&= \frac{e^{\beta(-1+i)}}{8-8i} = \frac{e^{\beta(-1+i)}(1+i)}{16}.
\end{aligned}$$

The integral can now be evaluated using the residue theorem. Due to closing the contour in the lower half plane, the contour has the reverse orientation, and hence an extra minus sign is required. Therefore

$$\begin{aligned}
\tilde{f}(\alpha) &= -2\pi i\left(\frac{\sqrt{2}}{\pi}\right)\left(\frac{e^{\beta(-1-i)}(-1+i)}{16}+\frac{e^{\beta(-1+i)}(1+i)}{16}\right)\\
&= \frac{e^{-\beta}\left(e^{-i\beta}+e^{i\beta}+i(-e^{i\beta}+e^{-i\beta})\right)}{4\sqrt{2}}\\
&= \frac{e^{-\beta}\left(\frac{e^{-i\beta}+e^{i\beta}}{2}+\frac{e^{i\beta}-e^{-i\beta}}{2i}\right)}{2\sqrt{2}}\\
&= \frac{e^{-\beta}(\cos\beta+\sin\beta)}{2\sqrt{2}}\\
&= \frac{e^{-\alpha/\sqrt{2}}\left(\cos\frac{\alpha}{\sqrt{2}}+\sin\frac{\alpha}{\sqrt{2}}\right)}{2\sqrt{2}}.
\end{aligned}$$

If $\alpha < 0$, then by using the substitution $x = -y$ it can be seen that

$$\tilde{f}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha x}dx}{x^4+1} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha(-y)}dy}{(-y)^4+1} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i(-\alpha)y)}dy}{y^4+1} = \tilde{f}(-\alpha)$$

and hence the Fourier transform is even. Hence, in general,

$$\tilde{f}(\alpha) = \frac{e^{-|\alpha|/\sqrt{2}}\left(\cos\frac{\alpha}{\sqrt{2}}+\sin\frac{|\alpha|}{\sqrt{2}}\right)}{2\sqrt{2}}.$$

4. The Laurent series of the integrand at $z = 0$ is

$$\frac{1-\cos x}{x^2} = \frac{1}{x^2}\left(1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\right) = \frac{1}{x^2}\left(\frac{x^2}{2}-\frac{x^4}{24}\ldots\right) = \frac{1}{2}-\frac{x^2}{24}+\ldots.$$

Since there is no term with a negative power, the point $z = 0$ is a removeable singularity, and hence the integrand can be extended to an analytic function there. Let J be an arbitrary contour from $-\infty$ to ∞ that bends below $z = 0$; since the integrand is analytic, the integral along this contour will be the same as along the real line. Then

$$\begin{aligned}
I &= \int_J \frac{1-\cos z}{z^2}dz\\
&= \int_J \frac{2-e^{iz}-e^{-iz}}{2z^2}dz\\
&= \int_J \frac{dz}{z^2} - \int_J \frac{e^{iz}}{2z^2}dz - \int_J \frac{e^{-iz}}{2z^2}dz.
\end{aligned}$$

All three integrals feature a pole of order 2 at $z = 0$. The first and third integrals can be closed by using large semicircles in the lower half plane; no poles will be enclosed and thus these integrals will be zero. The second integral must be closed in the upper half plane, and thus the pole at $z = 0$ will contribute. Note that

$$\frac{e^{iz}}{2z^2} = \frac{1}{2z^2}\left(1 + iz + \frac{(iz)^2}{2!} + \dots\right) = \frac{1}{2z^2} + \frac{i}{2z} - \frac{1}{4} + \dots$$

and thus the residue is $\frac{i}{2}$. Hence

$$I = 0 - 2\pi i\left(\frac{i}{2}\right) - 0 = \pi.$$

5. The integration can be carried out using a keyhole contour. A branch cut along the positive real axis can be introduced, and $\sqrt{z}$ can be taken to be positive and real above the cut. The integral around a circular contour of radius r is given by

$$\int_0^{2\pi} \frac{\sqrt{r}e^{i\theta/2}re^{i\theta}i\,d\theta}{1 + r^2e^{2i\theta}} = i\int_0^{2\pi} \frac{e^{3i\theta/2}r^{3/2}\,d\theta}{1 + r^2e^{2i\theta}}$$

This will decay to zero as $r \to \infty$ due to the presence of the r^2 term in the denominator. It will also decay to zero as $r \to 0$ due to the $r^{3/2}$ term in the numerator. Thus the only two components on the keyhole contour are the integral along the real axis above and below the branch cut. The integral above the contour is given by

$$I = \int_0^{\infty} \frac{\sqrt{r}}{1 + r^2}dr$$

and the integral below the contour is given by

$$J = -\int_0^{\infty} \frac{\sqrt{re^{2i\pi}}}{1 + r^2e^{4i\pi}}dr = -e^{i\pi}\int_0^{\infty} \frac{\sqrt{r}}{1 + r^2}dr = I.$$

The integrand has two simple poles at $\pm i$, and the keyhole contour will enclose both of these. The residue at $z = i$ is given by

$$\text{Res}\left(\frac{\sqrt{z}}{1 + z^2}, z = i\right) = \lim_{z \to i} \frac{\sqrt{z}(z - i)}{(z + i)(z - i)} = \frac{e^{i\pi/4}}{2i} = \frac{1 + i}{2\sqrt{2}i}$$

and the residue at $z = -i$ is given by

$$\text{Res}\left(\frac{\sqrt{z}}{1 + z^2}, z = -i\right) = \lim_{z \to -i} \frac{\sqrt{z}(z + i)}{(z + i)(z - i)} = \frac{e^{3i\pi/4}}{-2i} = \frac{1 - i}{2\sqrt{2}i}.$$

Hence, by using the residue theorem,

$$I + J = 2\pi i\left(\frac{1 + i}{2\sqrt{2}i} + \frac{1 - i}{2\sqrt{2}i}\right) = \frac{\pi(1 + i + 1 - i)}{\sqrt{2}} = \pi\sqrt{2}$$

and since $I = J$,

$$\int_0^{\infty} \frac{\sqrt{x}}{1 + x^2}dx = \frac{\pi}{\sqrt{2}}.$$

6. (a) Consider the integral

$$\oint_C \frac{e^{px}}{1+e^x}dx$$

where C is a closed rectangular contour with vertices at $\pm A$ and $\pm A + 2i\pi$. The integrand has singularities at $0 = 1 + e^x$ which corresponds to $x = (2n+1)i\pi$ where n is an integer. Hence the contour encloses one singularity at $x = i\pi$. By writing $x = i\pi + y$, it can be seen that for small y,

$$\begin{aligned}
\frac{e^{px}}{1+e^x} &= \frac{e^{pi\pi}e^{py}}{1+e^{i\pi+y}} \\
&= \frac{e^{i\pi p}\left(1+py+\frac{p^2y^2}{2}+\dots\right)}{1-e^y} \\
&= \frac{e^{i\pi p}\left(1+py+\frac{p^2y^2}{2}+\dots\right)}{1-\left(1+y+\frac{y^2}{2}+\dots\right)} \\
&= -\frac{e^{i\pi p}\left(1+py+\frac{p^2y^2}{2}+\dots\right)}{y+\frac{y^2}{2}+\dots} \\
&= -\frac{e^{i\pi p}}{y}\left(1+py+\frac{p^2y^2}{2}+\dots\right)\left(1-\frac{y}{2}+\dots\right)
\end{aligned}$$

and hence the residue at $i\pi$ is $-e^{i\pi p}$. Now consider the integral around C. The integration from A to $A + 2i\pi$ is given by

$$\int_0^{2\pi} \frac{e^{p(A+iy)}}{1+e^{A+iy}}idy$$

and this will vanish in the limit as $A \to \infty$, since the e^A in the denominator will dominate over the e^{pA} in the numerator (since $p < 1$). Similarly, the integration from $-A + 2i\pi$ to $-A$ is given by

$$-\int_0^{2\pi} \frac{e^{p(-A+iy)}}{1+e^{-A+iy}}idy$$

and this will also vanish due to the e^{-Ap} term in the numerator. In the limit as $A \to \infty$, the two contributions will come from the integral along the real line,

$$I = \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}dx,$$

and the integral from $2\pi i - \infty$ to $2\pi i + \infty$,

$$J = -\int_{-\infty}^{\infty} \frac{e^{p(x+2\pi i)}}{1+e^{x+2\pi i}}dx = -e^{2\pi ip}\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x}dx = -e^{2\pi ip}.$$

By using the residue theorem

$$I + J = 2\pi i \left(-e^{i\pi p}\right)$$

and therefore

$$I(1 - e^{2\pi i p}) = -2\pi i e^{i\pi p}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x} dx = I = \frac{2\pi i e^{i\pi p}}{e^{2\pi i p} - 1} = \frac{2i\pi}{e^{\pi i p} - e^{-\pi i p}} = \frac{\pi}{\sin \pi p}.$$

(b) By making the substitution $y = e^x$, so that $dy/y = dx$, the integral can be written as

$$\int_0^{\infty} \frac{y^p \, dy}{(1+y)y} = \int_0^{\infty} \frac{y^{p-1}}{(1+y)} dy.$$

One definition of the beta function is

$$\mathrm{B}(p,q) = \int_0^{\infty} \frac{y^{p-1} \, dy}{(1+y)^{p+q}}$$

and it can be seen that this matches the given expression for $q = 1 - p$, and therefore

$$\mathrm{B}(1-p,p) = \int_0^{\infty} \frac{y^{p-1} \, dy}{(1+y)} = \frac{\pi}{\sin \pi p}.$$

Using the identity linking beta functions and gamma functions, it can be seen that

$$\frac{\pi}{\sin \pi p} = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma\left(p + (1-p)\right)} = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \Gamma(p)\Gamma(1-p).$$

7. Since the integrand is even, the integral can be written as

$$\int_0^{\infty} \frac{x \, dx}{\sinh x} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \, dx}{\sinh x}.$$

The hyperbolic sine function can be written in terms of the sine function as

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{-i(ix)} - e^{i(ix)}}{2} = \frac{e^{i(ix)} - e^{-i(ix)}}{2i(i)} = \frac{\sin ix}{i}.$$

The zeros of $\sin x$ are at $x = n\pi$ for any integer n, and hence the zeros of $\sinh z$ are at $z = in\pi$. Hence $1/\sinh z$ has simple poles at $z = in\pi$. The function $z/\sinh z$ will also have simple poles at these locations, with the possible exception at $z = 0$, where the function has a Laurent series

$$\frac{z}{\sinh z} = \frac{z}{z + \frac{z^3}{3!} + \ldots} = \frac{1}{1 + \frac{z^2}{6} + \ldots} = 1 - \frac{z^2}{6} + \ldots.$$

Since there is no term with a negative power in this expression, $z = 0$ is a removeable singularity. Consider integrating around the rectangle with corners at $\pm R$ and $\pm R + i\pi$. The integration on the line from R to $R + i\pi$ will be

$$\int_0^\pi \frac{(R+iy)idy}{e^{R+iy} - e^{-R-iy}}$$

which will vanish as $R \to \infty$ due to the presence of the e^R term in the denominator. A similar argument can be used to determine that the integral along $-R$ to $-R + i\pi$ will also vanish in the limit. The integral from $R + i\pi$ to $R - i\pi$ will go over the simple pole at $z = i\pi$. To evaluate the residue at this point, write $z = y + i\pi$, and hence

$$\begin{aligned}
\frac{z}{\sinh z} &= \frac{y+i\pi}{\sinh(y+i\pi)} = \frac{y+i\pi}{-\sinh y} \\
&= \frac{y+i\pi}{-y+\frac{y^3}{6}-\dots} = \frac{y+i\pi}{y\left(1-\frac{y^2}{6}+\dots\right)} \\
&= \frac{-1}{y}(i\pi+y)\left(1+\frac{y^2}{6}-\dots\right) \\
&= \frac{-i\pi}{y}+\dots
\end{aligned}$$

where the identity $\sinh(y + i\pi) = -\sinh y$ has been employed. Hence the residue is $-i\pi$. Since residues on the boundary of a contour give a half contribution it follows that

$$\begin{aligned}
2\pi i\left(\frac{-i\pi}{2}\right) &= \int_{-\infty}^{\infty}\frac{x\,dx}{\sinh x} - \int_{-\infty}^{\infty}\frac{(x+i\pi)\,dx}{\sinh(x+i\pi)} \\
&= \int_{-\infty}^{\infty}\frac{x\,dx}{\sinh x} - \int_{-\infty}^{\infty}\frac{(x+i\pi)\,dx}{-\sinh x} \\
&= \int_{-\infty}^{\infty}\frac{x\,dx}{\sinh x} + \int_{-\infty}^{\infty}\frac{x\,dx}{-\sinh x} + i\pi\int_{-\infty}^{\infty}\frac{dx}{\sinh x}.
\end{aligned}$$

The first two integrals are identical, and the final integral can be neglected since the integrand is odd. Hence

$$\int_{-\infty}^{\infty}\frac{x\,dx}{\sinh x} = \frac{\pi^2}{2}$$

and therefore

$$\int_0^{\infty}\frac{x\,dx}{\sinh x} = \frac{\pi^2}{4}.$$

An alternative approach is to integrate around the rectangle with corners at $\pm R$ and $\pm R + \frac{i\pi}{2}$. This contour encloses no singularities so

$$0 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{z\,dz}{\sinh z} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{(x+\frac{i\pi}{2})dx}{\sinh\left(x+\frac{i\pi}{2}\right)}$$

and hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{z\,dz}{\sinh z} &= \int_{-\infty}^{\infty} \frac{(x+\frac{i\pi}{2})dx}{\sinh\left(x+\frac{i\pi}{2}\right)} \\
&= \int_{-\infty}^{\infty} \frac{(x+\frac{i\pi}{2})dx}{i\cosh x} \\
&= \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{dx}{\cosh x}
\end{aligned}$$

where the integration of the $\frac{x}{\cosh x}$ term can be neglected since this function is odd. This can be written as

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{z\,dz}{\sinh z} &= \pi\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} \\
&= \pi\int_{-\infty}^{\infty} \frac{e^x\,dx}{1+e^{2x}} \\
&= \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{e^{y/2}\,dy}{1+e^y} \\
&= \frac{\pi}{2}\left(\frac{\pi}{\sin\frac{\pi}{2}}\right) = \frac{\pi^2}{4}
\end{aligned}$$

by making use of the result from part (a) of the previous exercise. Hence

$$\int_0^{\infty} \frac{x\,dx}{\sinh x} = \frac{\pi^2}{4}.$$

8. By using the Bromwich inversion integral, the inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{pe^{pt}\,dp}{(p+1)(p^2+4)}$$

where the constant c is chosen so that the contour lies to the right of all of the poles of the integrand. The integrand can be written as

$$\frac{pe^{pt}}{(p+1)(p+2i)(p-2i)}$$

and thus has simple poles at $p=-1$ and $p=\pm 2i$. The residues at $p=1$ is given by

$$\begin{aligned}
\text{Res}\left(\frac{pe^{pt}}{(p+1)(p^2+4)}, p=-1\right) &= \lim_{p\to -1}\frac{p(p+1)e^{pt}}{(p+1)(p^2+4)} \\
&= \frac{-e^{-t}}{1+4} = -\frac{e^{-t}}{5}.
\end{aligned}$$

The residues at $p = \pm 2i$ are given by

$$\begin{aligned}
\text{Res}\left(\frac{pe^{pt}}{(p+1)(p^2+4)}, p = \pm 2i\right) &= \lim_{p\to\pm 2i} \frac{p(p\mp 2i)e^{pt}}{(p+1)(p+2i)(p-2i)} \\
&= \lim_{p\to\pm 2i} \frac{pe^{pt}}{(p+1)(p\pm 2i)} \\
&= \frac{e^{\pm 2it}}{2(1\pm 2i)} \\
&= \frac{e^{\pm 2it}(1\mp 2i)}{10}.
\end{aligned}$$

Hence, by closing the integration with a large semicircle in the left half plane, which will enclose all three poles within the contour,

$$\begin{aligned}
f(t) &= \frac{2\pi i}{2\pi i}\left(-\frac{e^{-t}}{5} + \frac{e^{2it}(1-2i)}{10} + \frac{e^{2it}(1+2i)}{10}\right) \\
&= \frac{1}{5}\left(-e^{-t} + \frac{1}{2}\left(e^{2it} + e^{-2it}\right) + \frac{2}{2i}\left(e^{2it} - e^{-2it}\right)\right) \\
&= \frac{-e^{-t} + \cos 2t + 2\sin 2t}{5}.
\end{aligned}$$