Math 121A: Homework 1 solutions

1. Let a_k be the amount of the impurity removed at the kth stage. Then $a_1 = 1/n$, and $a_{k+1} = a_k/n$, so $a_{k+1} = n^{-(k+1)}$. The total amount of impurity removed is

$$T = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{n^k}.$$

This is a geometric series, and hence

$$T = \frac{\frac{1}{n}}{1-\frac{1}{n}} = \frac{1}{n-1}$$

If $n = 2$, then $T = 1$ and all of the impurity will be removed. If $n = 3$, then $T = 1/2$ and at least of half of the impurity will remain.

2. Since 2^n grows much more rapidly than n^2, it should be expected that $\lim_{n\to\infty} 2^n/n^2 = 0$. However to see this explicity, note that for $n \geq 3$, the binomial expansion can be used to see that

$$\begin{aligned} 2^n &= (1+1)^n \\ &= 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} + \ldots \\ &> \frac{n(n-1)(n-2)}{6} \\ &> \frac{(n-2)^3}{6} \end{aligned}$$

Since

$$\lim_{n\to\infty} \frac{(n-2)^3}{n^2} = \infty$$

it follows that $\lim_{n\to\infty} 2^n/n^2 = \infty$.

3. The partial sum of the first N terms is

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= \frac{1}{1} - \frac{1}{N+1}. \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1.$$

4. (a) Initially, the car and bee are moving toward each other at speed $b + c$, and will reach each other at $t = d/(b + c)$. After this time, the bee will have travelled

$$a_1 = \frac{bd}{b + c}. \tag{1}$$

Since the bee travels the same distance back toward Q, it must be the case that $a_2 = a_1$. To determine a_3, first note that the bee will reach Q for the second time at time

$$t' = \frac{2d}{b + c}.$$

The distance from the car from the point Q at this time is therefore

$$d' = d - ct' = d - \frac{2cd}{b + c} = d\frac{b - c}{b + c}.$$

The problem is now equivalent to the original configuration, except that the car is at a distance d' from Q instead of d, and thus by reference to Eq. 1 it can be seen that

$$a_3 = \frac{bd'}{b + c} = \frac{bd}{b + c}\left(\frac{b - c}{b + c}\right)$$

Again, $a_4 = a_3$. To determine a_5, note that the car is now a distance

$$d'' = d\left(\frac{b - c}{b + c}\right)^2$$

so that another factor of $(b - c)/(b + c)$ will be introduced each time the bee carries out a zig-zag. In general, therefore, the distances are given by

$$a_{2n} = a_{2n-1} = \frac{bd}{b + c}\left(\frac{b - c}{b + c}\right)^{n-1}.$$

(b) The sum of the series is

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= 2\sum_{n=0}^{\infty} a_{2n} \\
&= 2\sum_{n=0}^{\infty} \frac{bd}{b + c}\left(\frac{b - c}{b + c}\right)^{n-1} \\
&= 2\frac{bd}{b + c}\frac{1}{1 - \frac{b-c}{b+c}} \\
&= \frac{2bd}{(b + c) - (b - c)} \\
&= \frac{bd}{c}.
\end{aligned}$$

This result should be expected. The car reaches Q at time $t_f = d/c$. Since the bee is moving with constant speed c during this process, it will cover a distance of bd/c in this interval.

(c) Now consider the case when the bee moves at a speed b' when moving toward the car. Initially, the car and the bee are moving toward each other with speed $b' + c$, and will reach each other at $t = d/(b' + c)$. Again, $a_2 = a_1$. The bee will reach Q for a second time at time

$$t' = \frac{d}{b+c} + \frac{d}{b'+c}.$$

This distance of the car from the point Q at this time is therefore

$$\begin{aligned} d' &= d - ct' \\ &= d - \frac{cd}{b'+c} - \frac{cd}{b+c} \\ &= d\left(\frac{(b+c)(b'+c) - c(b'+c) - c(b+c)}{(b'+c)(b+c)}\right) \\ &= d\left(\frac{bb' - c^2}{(b'+c)(b+c)}\right). \end{aligned}$$

Using the same logic as previously, it follows that

$$a_{2n} = a_{2n-1} = \frac{bd}{b'+c}\left(\frac{bb'-c^2}{(b'+c)(b+c)}\right)^{n-1}$$

The total distance covered by the bee is

$$\begin{aligned} \sum_{n=0}^{\infty} a_n &= 2\sum_{n=0}^{\infty} a_{2n} \\ &= 2\sum_{n=0}^{\infty} \frac{bd}{b+c}\left(\frac{bb'-c^2}{(b'+c)(b+c)}\right)^{n-1} \\ &= 2\frac{bd}{b+c}\frac{1}{1-\frac{bb'-c^2}{(b'+c)(b+c)}} \\ &= \frac{2bd(b+c)}{(b'+b+2c)c}. \end{aligned}$$

If $b = b'$, this agress with the result from part (b).

5. For all natural numbers n, note that

$$n + \frac{1}{2} < 2n$$

and hence

$$\frac{1}{n+\frac{1}{2}} > \frac{1}{2n}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

diverges, it follows that $\sum_{n=1}^{\infty} 1/(n+\frac{1}{2})$ diverges via the comparison test.

6. To determine the convergence properties of

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4},$$

consider the integral

$$\int^{\infty} \frac{x}{x^2+4} dx = \left[\frac{1}{2}\log(x^2+4)\right]^{\infty}$$

and since $\log x$ increases without limit, the integral diverges. Hence the series diverges by the integral test.

7. The coefficients of the series are $a_n = e^n/\sqrt{n!}$. Hence, using the ratio test,

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{e^{n+1}}{\sqrt{(n+1)!}} \frac{\sqrt{n!}}{e^n}\right| = \left|\frac{e}{\sqrt{n+1}}\right| = \frac{e}{\sqrt{n+1}}$$

and hence

$$\lim_{n\to\infty} \rho_n = 0$$

so the series converges.

8. For the given series, $a_n = (\log n)^{-1}$. Note that $|a_{n+1}| \leq |a_n|$ and $\lim_{n\to\infty} a_n = 0$ and hence it must converge by the alternating series theorem.

9. By using the standard test,

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2}{3}\right| = \frac{2}{3}.$$

and thus the radius of convergence is $R = 1/\rho = 3/2$. If $x = 3/2$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(2^3/2)^n}{3^n} = \sum_{n=0}^{\infty} (1)^n$$

which increases without limit and hence diverges. If $x = -3/2$, the series becomes

$$\sum_{n=0}^{\infty} (-1)^n$$

which also diverges. Hence the exact interval of convergence is $-3/2 < x < 3/2$.

10. (a) By using the standard test,

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$$

and hence the radius of convergence is $R = 1$. If $x = 1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

which diverges by the integral test, and if $x = -1$ the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

which converges by the alternating series test. Hence the exact interval of convergence is $-1 \leq x < 1$.

(b) Show that the power series

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{y}{1+y^2} \right)^n$$

converges for all real numbers y. If $|y| < 1$, then

$$\left| \frac{y}{1+y^2} \right| \leq |y| < 1$$

and if $|y| \geq 1$ then

$$\left| \frac{y}{1+y^2} \right| < \frac{|y|}{|y|^2} = \frac{1}{|y|} \leq 1.$$

Therefore if $x = y/(1+y^2)$, then $|x| < 1$ for any choice of y, and thus

$$\sum_{n=0}^{\infty} \frac{1}{n} \left(\frac{y}{1+y^2} \right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n}$$

will converge, since x lies within the interval of convergence.