Solutions to sample final questions

1. To construct a Green function solution, consider finding a solution to the equation

$$y'' + k^2 y = \delta(x - x')$$

where x' is a parameter. In the regions $x < x'$ and $x > x'$

$$y'' + k^2 y = 0.$$

By searching for solutions of the form $y(x) = e^{mx}$ it can be seen that $m^2 + k^2 = 0$ and thus $m = \pm ik$. Hence for $x < x'$, the solution can be expressed as

$$y(x) = A \cos kx + B \sin kx$$

for some constants A and B. In the region $x > x'$ the solution will also have the same form, but potentially with different constants C and D:

$$y(x) = C \cos kx + D \sin kx.$$

Since $y(0) = 0$, it follows that $A = 0$, and since $y(\frac{\pi}{2k}) = 0$, then $D = 0$. To set the two remaining constants, consider $x = x'$. The function must be continuous there, and hence

$$B \sin kx' = C \cos kx'. \tag{1}$$

There must be jump in the derivative of 1 at $x = x'$, and hence

$$1 = (-Ck \sin kx') - Bk \cos kx'. \tag{2}$$

See appendix A for a derivation of this condition. By substituting Eq. 1 into Eq. 2 gives

$$\sin kx' = -Ck \sin^2 kx' - Ck \cos^2 kx' = -Ck$$

and hence

$$C = -\frac{\sin kx'}{k}.$$

From Eq. 1 it follows that

$$B = -\frac{\cos kx'}{k}.$$

Thus the Green function solution is

$$G(x, x') = \begin{cases} \frac{-\cos kx' \sin kx}{k} & \text{for } x < x', \\ \frac{-\sin kx' \cos kx}{k} & \text{for } x > x'. \end{cases}$$

Plots of $G(x, x')$ are shown in Fig. 1 for the cases of $kx' = \pi/12, \pi/6, \pi/4, \pi/3, 5\pi/12$.

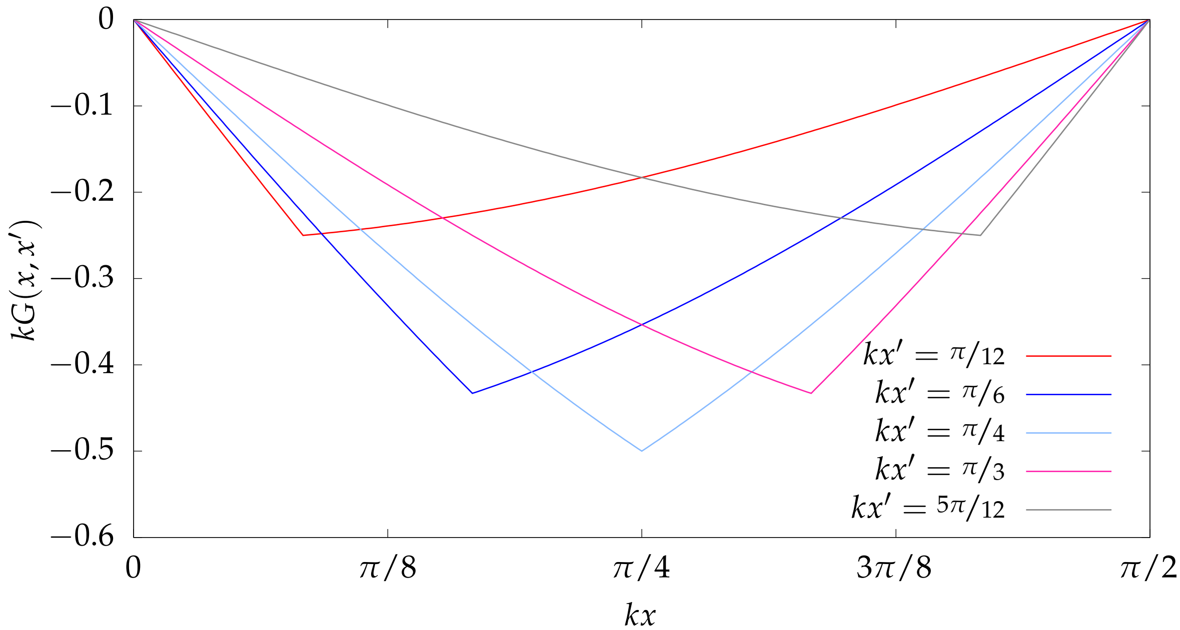

Figure 1: Plots of several Green functions corresponding to different values of x'.

2. (a) If $z = x + iy$ and $r = a + ib$ then

$$\begin{aligned} 0 = \bar{r}z - r\bar{z} &= (a - ib)(x + iy) - (a + ib)(x - iy) \\ &= (ax + by + iay - ibx) - (ax + by - iay + ibx) \\ &= 2i(ay - bx) \end{aligned}$$

and this will be satisfied if $ay = bx$, which can be written as $y = (b/a)x$. This is a straight line passing through (a, b).

(b) If $z = x + iy$ then

$$\begin{aligned} z^2 + \bar{z}^2 &= (x + iy)^2 + (x - iy)^2 \\ &= (x^2 - y^2 + 2ixy) + (x^2 - y^2 - 2ixy) \\ &= 2(x^2 - y^2) \end{aligned}$$

and

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2.$$

Hence

$$\begin{aligned} (z^2 + \bar{z}^2)(b^2 - a^2) + 2z\bar{z}(b^2 + a^2) &= 2(x^2 - y^2)(b^2 - a^2) + 2(x^2 + y^2)(b^2 + a^2) \\ &= 2(x^2b^2 + a^2y^2 - x^2a^2 - y^2b^2) \\ &\quad + 2(x^2b^2 + a^2y^2 + x^2a^2 + y^2b^2) \\ &= 4(x^2b^2 + y^2a^2) \end{aligned}$$

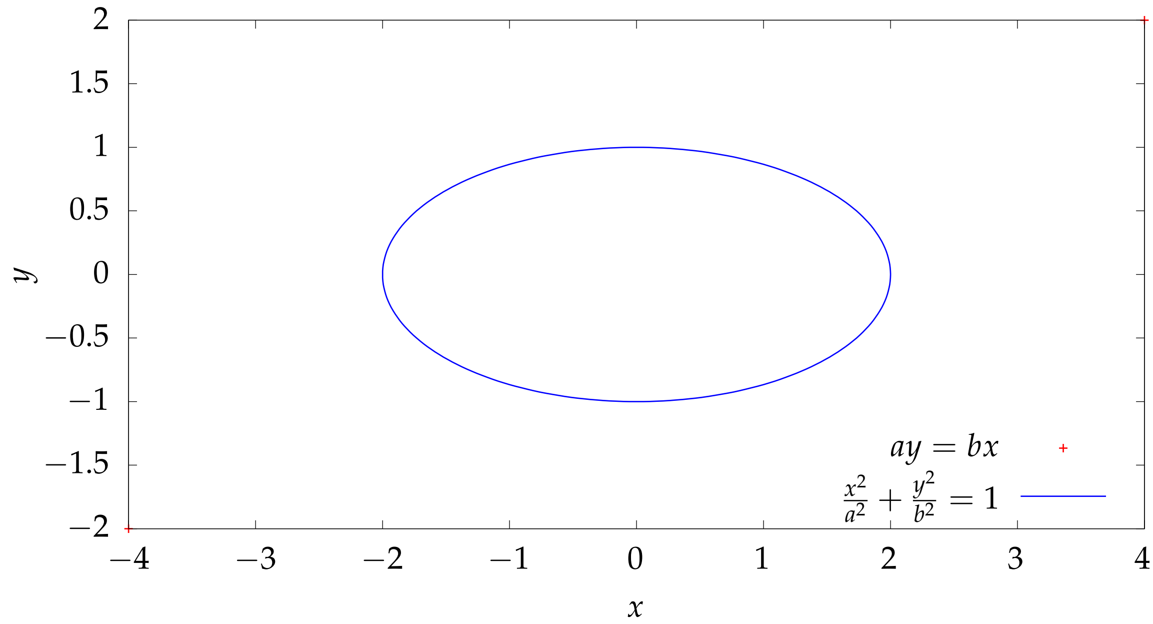

Figure 2: Plots of the loci of the two complex equations considered in question 2.

and therefore

$$4(x^2b^2 + y^2a^2) = 4a^2b^2.$$

Dividing by $4a^2b^2$ gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Plots of the two loci are shown in Fig. 2.

3. The volume of the box is given by

$$V(d, e, f) = 8def$$

and the constraint is

$$\frac{d^2}{a^2} + \frac{e^2}{b^2} + \frac{f^2}{c^2} \leq 1.$$

The given constraint describes an ellipsoid. If the maximum of V was in the interior of the ellipsoid, then

$$0 = \frac{\partial V}{\partial d} = 8ef, \qquad 0 = \frac{\partial V}{\partial e} = 8df, \qquad \frac{\partial V}{\partial f} = 8de.$$

These equations are only satisfied if at least two of d, e, and f are zero, and the corresponding volume will be $V = 0$, which is not a maximum. Hence the maximum volume must correspond to d, e, and f being on the surface of the ellipsoid.

To use the method of Lagrange multipliers, consider minimizing the augmented function

$$V(d,e,f,\lambda) = 8def + \lambda\left(1 - \frac{d^2}{a^2} - \frac{e^2}{b^2} - \frac{f^2}{c^2}\right).$$

Then

$$\frac{\partial V}{\partial d} = 8ef - \frac{2\lambda d}{a^2} = 0, \qquad \frac{\partial V}{\partial e} = 8df - \frac{2\lambda e}{b^2} = 0, \qquad \frac{\partial V}{\partial f} = 8de - \frac{2\lambda f}{c^2} = 0.$$

Hence

$$\frac{\lambda}{4} = \frac{a^2 ef}{d} = \frac{b^2 df}{e} = \frac{c^2 de}{f}$$

and therefore

$$\frac{a^2}{d^2} = \frac{b^2}{e^2} = \frac{c^2}{f^2}. \tag{3}$$

The constraint equation therefore gives

$$1 = \frac{d^2}{a^2} + \frac{e^2}{b^2} + \frac{f^2}{c^2} = \frac{3d^2}{a^2}$$

and thus

$$d = \frac{a}{\sqrt{3}}$$

after which it follows from Eq. 3 that

$$e = \frac{b}{\sqrt{3}}, \qquad f = \frac{c}{\sqrt{3}}.$$

The volume is

$$V(d,e,f) = \frac{8abc}{3\sqrt{3}}.$$

4. (a) The eigenvalues are solutions to

$$0 = \det|A - \lambda I| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 4 = \lambda(\lambda - 5).$$

Hence 0 and 5 are eigenvalues. For $\lambda = 0$, an eigenvector will satisfy

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and hence $(u, v) = (-1, 2)$ is a solution. For $\lambda = 5$, an eigenvector will satisfy

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and thus a solution is $(u, v) = (2, 1)$. It should be noted that the two eigenvectors are orthogonal, as would be expected for a symmetric matrix.

(b) The product of A with itself is

$$A^2 = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 10 \\ 10 & 5 \end{pmatrix} = 5A$$

and thus $\beta = 5$. For the given identity, $A^n = 5^{n-1}A$, it can be seen that the case of $n = 1$ is trivially satisfied. Now suppose that the case of n is true, and consider the case of $n + 1$. Then

$$A^{n+1} = A^2 A^{n-1} = (5A)A^{n-1} = 5A^n = 5(5^{n-1}A) = 5^{(n+1)-1}A$$

and the result is true for $n + 1$. Hence, by mathematical induction, the result must be true for all n.

(c) The exponential is

$$\begin{aligned} \exp(\lambda A) &= \sum_{n=0}^{\infty} \frac{(\lambda A)^n}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{(\lambda A)^n}{n!} \\ &= I + \sum_{n=1}^{\infty} \frac{\lambda^n 5^{n-1} A}{n!} \\ &= I + \frac{A}{5} \sum_{n=1}^{\infty} \frac{5^n \lambda^n}{n!} \\ &= I + \frac{A}{5} \left(-1 + \sum_{n=0}^{\infty} \frac{5^n \lambda^n}{n!} \right) = I + \left(\frac{e^{5\lambda} - 1}{5} \right) A \end{aligned}$$

and thus $f(\lambda) = 1$ and $g(\lambda) = (e^{5\lambda} - 1)/5$.

5. (a) If $f(x) = \log(1 - x)$, then $f(0) = 0$. The first derivative of $f(x) = \log(1 - x)$ is

$$f'(x) = -\frac{1}{1 - x}$$

and thus $f'(0) = -1$. The successive derivatives are

$$f''(x) = -\frac{1}{(1 - x)^2}, \qquad f'''(x) = -\frac{2}{(1 - x)^3}, \qquad f^{(4)}(x) = -\frac{2 \cdot 3}{(1 - x)^4}.$$

It can be seen that each successive derivative brings an additional power of $(1-x)$ and an integer coefficient, and a therefore a general expression is

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}$$

for any positive integer n, and hence $f^{(n)}(0) = -(n-1)!$. The Taylor series is therefore

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!} = \sum_{n=1}^{\infty} \frac{-(n-1)!x^n}{n!} = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

A comparison between the function and the first few terms of its Taylor series is shown in Fig. 3(a). If the series is written as $\sum a_n x^n$, then the radius of convergence R of this power series is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1.$$

Hence $R = 1$. To find the exact interval of convergence, consider the end points at $x = \pm R$. At $x = 1$, the series is

$$-\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series and diverges. At $x = -1$, the series is

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series theorem. Hence the exact interval of convergence is $-1 \le x < 1$.

(b) The power series is given by

$$\begin{aligned}
\log(1-x^2-x^3) &= -(x^2+x^3) - \frac{(x^2+x^3)^2}{2} - \frac{(x^2+x^3)^3}{3} - \ldots \\
&= -x^2 - x^3 - \frac{x^4}{2} - \frac{2x^5}{2} - \frac{x^6}{2} - \frac{x^6}{3} - \ldots \\
&= -x^2 - x^3 - \frac{x^4}{2} - x^5 - \frac{5x^6}{6} - \ldots
\end{aligned}$$

A comparison between the function and the Taylor series is shown in Fig. 3(b).

6. The integral can be evaluated by considering a keyhole contour as shown in Fig. 4, consisting of four components A, B, C, and D. A branch cut can be introduced along the positive real axis, where the value of $z^{1/3}$ is taken to be positive and real just

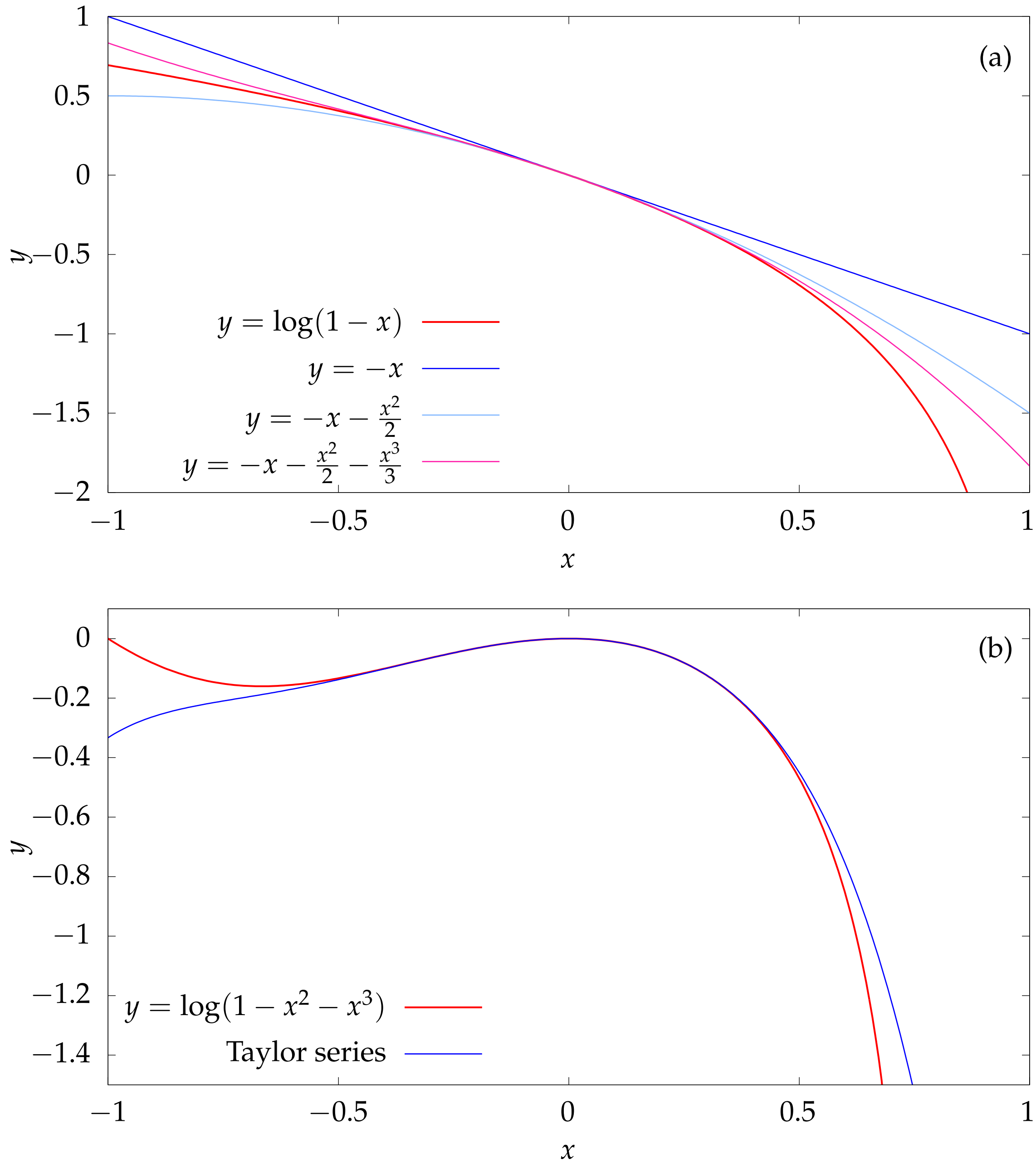

Figure 3: (a) Comparison of $\log(1-x)$ to its first, second, and third order Taylor series. (b) Comparison of $\log(1-x^2-x^3)$ to its Taylor series up to terms in x^6.

above the cut. An integral around a circular contour given by $z = \rho e^{i\theta}$ for $0 \le \theta < 2\pi$ is

$$I(\rho) = \int_0^{2\pi} \frac{\rho^{1/3} e^{i\theta/3} \rho e^{i\theta} i d\theta}{(\rho e^{i\theta} + 2)^2}.$$

It can be seen that $I(\rho) \to 0$ as $\rho \to \infty$, because the ρ^2 factor in the denominator will dominate. In addition $I(\rho) \to 0$ as $\rho \to 0$ as the $\rho^{4/3}$ factor in the numerator will tend to zero. In the limit, only the integrals I_A and I_C along the contours A and C will matter. It can be seen that

$$I_C = \int_C \frac{z^{1/3} dz}{(z+2)^2} = -\int_0^\infty \frac{r^{1/3} e^{2\pi i/3} dr}{(re^{2\pi i} + 2)^2} = -\int_0^\infty \frac{r^{1/3} e^{2\pi i/3} dr}{(r+2)^2} = -e^{2\pi i/3} I_A.$$

By the residue theorem, the integral around the entire contour will be determined by the residue at the enclosed pole of order 2 at $z = -2$. The residue is given by

$$\begin{aligned} \text{Res}\left(\frac{z^{1/3}}{(z+2)^2}, z = -2 \right) &= \lim_{z \to -2} \frac{d}{dz} \left(\frac{z^{1/3}(z+2)^2}{(z+2)^2} \right) \\ &= \lim_{z \to -2} \frac{d}{dz} z^{1/3} \\ &= \lim_{z \to -2} \frac{z^{-2/3}}{3} \\ &= \frac{2^{-2/3} e^{-2i\pi/3}}{3}. \end{aligned}$$

Hence

$$I_A + I_C = 2\pi i \frac{e^{-2i\pi/3}}{2^{2/3} \cdot 3}$$

and therefore

$$I_A(1 - e^{2\pi i/3}) = 2\pi i \frac{e^{-2i\pi/3}}{2^{2/3} \cdot 3}.$$

By making use of $e^{-i\pi} = -1$, it can be seen that

$$\begin{aligned} I_A &= 2\pi i \frac{e^{-2i\pi/3}}{2^{2/3}(1 - e^{2\pi i/3}) \cdot 3} \\ &= \frac{2\pi i e^{-i\pi}}{2^{2/3} \cdot 3(e^{-\pi i/3} - e^{\pi i/3})} \\ &= \frac{\pi}{2^{2/3} \cdot 3 \sin \frac{\pi}{3}} \\ &= \frac{\pi}{2^{2/3} \cdot 3 \frac{\sqrt{3}}{2}} = \frac{2^{1/3}\pi}{3^{3/2}}. \end{aligned}$$

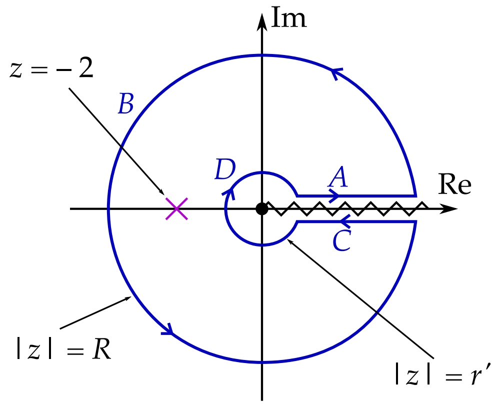

Figure 4: Keyhole contour considered in question 4. The integrand has a pole of order 2 at $z = -2$, and a branch cut can be introduced along the positive real axis. A closed contour can be constructed as (A) a section from r' to R above the cut, (B) the circle $|z| = R$, (C) a section from R to r' below the cut, and (D) the circle $|z| = r'$.

7. By using the chain rule

$$\frac{\partial f}{\partial a} = \frac{\partial x}{\partial a}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial a}\frac{\partial f}{\partial y} = b\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} \tag{4}$$

and

$$\frac{\partial f}{\partial b} = \frac{\partial x}{\partial b}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial b}\frac{\partial f}{\partial y} = a\frac{\partial f}{\partial x} - b\frac{\partial f}{\partial y}. \tag{5}$$

Taking Eq. 4 multiplied b, plus Eq. 5 multiplied by a, gives

$$b\frac{\partial f}{\partial a} + a\frac{\partial f}{\partial b} = (b^2 + a^2)\frac{\partial f}{\partial x} + (ab - ab)\frac{\partial f}{\partial y}$$

and hence

$$\frac{\partial f}{\partial x} = \frac{b}{b^2 + a^2}\frac{\partial f}{\partial a} + \frac{a}{b^2 + a^2}\frac{\partial f}{\partial b}$$

Similarly

$$\frac{\partial f}{\partial y} = \frac{a}{a^2 + b^2}\frac{\partial f}{\partial a} - \frac{b}{a^2 + b^2}\frac{\partial f}{\partial b}.$$

8. (a) The Laplace transform of f' is given by

$$\begin{aligned} L(f')(p) &= \int_0^\infty f'(t)e^{-pt}dt \\ &= \left[f(t)e^{-pt}\right]_0^\infty - \int_0^\infty f(t)(-pe^{-pt})dt \\ &= pF(p) - f(0). \end{aligned}$$

By applying this result recursively, it can be seen that the Laplace transform of f'' is

$$\begin{aligned} L(f'')(p) &= pL(f')(p) - f'(0) \\ &= p(pF(p) - f(0)) - f'(0) \\ &= p^2F(p) - pf(0) - f'(0). \end{aligned}$$

(b) Taking the Laplace transform of the equation gives

$$(p^2F(p) - pf(0) - f'(0)) + 9(pF(p) - f(0)) + 8F(p) = 0$$

which can be rearranged to give

$$(p^2 + 9p + 8)F(p) - p - 1 - 9 = 0$$

and hence

$$F(p) = \frac{p+10}{(p+8)(p+1)}.$$

(c) The Bromwich inversion integral is

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}(p+10)dp}{(p+8)(p+1)}$$

where $c > 0$. A closed contour can be made by making use of a large semicircle in the left half plane. The integrand has two simple poles at $p = -1$ and $p = -8$, both of which will be enclosed by the contour The residues at these poles are

$$\text{Res}\left(\frac{e^{pt}(p+10)}{(p+8)(p+1)}, p = -1\right) = \lim_{p\to -1} \frac{e^{pt}(p+10)}{(p+8)} = \frac{9e^{-t}}{7}$$

and

$$\text{Res}\left(\frac{e^{pt}(p+10)}{(p+8)(p+1)}, p = -8\right) = \lim_{p\to -8} \frac{e^{pt}(p+10)}{(p+1)} = -\frac{2e^{-8t}}{7}.$$

Hence the integral is

$$f(t) = \frac{1}{2\pi i}2\pi i\left(\frac{9e^{-t}}{7} - \frac{2e^{-8t}}{7}\right) = \frac{9e^{-t} - 2e^{-8t}}{7}.$$

9. (a) The Fourier transform of $f'(x)$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{-ix\alpha}dx$$

and by using integration by parts this can be written as

$$\frac{1}{2\pi}\left[f(x)(-i\alpha)e^{-ix\alpha}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\alpha)e^{-ix\alpha}.$$

Since $f(x) \to 0$ as $x \to \pm\infty$ is a condition for using Fourier transforms, it follows that this is equal to

$$i\alpha \int_{-\infty}^{\infty} f(x)(-i\alpha)e^{-ix\alpha},$$

which is $i\alpha\tilde{f}(\alpha)$.

(b) The Fourier transform of the delta function is

$$\tilde{\delta}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{-i\alpha x}dx = \frac{e^{-i\alpha 0}}{2\pi} = \frac{1}{2\pi}.$$

Using this result, and the result from part (a), the Fourier transform of the differential equation is

$$(i\alpha)^2\tilde{f}(\alpha) + i\alpha\tilde{f}\alpha - 2\tilde{f} = \frac{1}{2\pi}$$

and hence

$$\tilde{f}(\alpha) = -\frac{1}{2\pi(\alpha^2 - i\alpha + 2)} = -\frac{1}{2\pi(\alpha - 2i)(\alpha + i)}.$$

(c) The solution is given by

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(\alpha)e^{i\alpha x}d\alpha = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}d\alpha}{(\alpha - 2i)(\alpha + i)} \tag{6}$$

The integrand has simple poles at $\alpha = 2i$ and $\alpha = -i$. The residues are

$$\begin{aligned} \text{Res}\left(\frac{e^{i\alpha x}}{(\alpha - 2i)(\alpha + i)}, \alpha = 2i\right) &= \lim_{\alpha \to 2i} \frac{e^{i\alpha x}(\alpha - 2i)}{(\alpha + i)(\alpha - 2i)} \\ &= \lim_{\alpha \to 2i} \frac{e^{i\alpha x}}{\alpha + i} = \frac{e^{-2x}}{3i} \end{aligned}$$

and

$$\begin{aligned} \text{Res}\left(\frac{e^{i\alpha x}}{(\alpha - 2i)(\alpha + i)}, \alpha = -i\right) &= \lim_{\alpha \to -i} \frac{e^{i\alpha x}(\alpha + i)}{(\alpha + i)(\alpha - 2i)} \\ &= \lim_{\alpha \to -i} \frac{e^{i\alpha x}}{\alpha - 2i} = -\frac{e^{x}}{3i}. \end{aligned}$$

Now consider the integral in Eq. 6. If $x \geq 0$, a closed contour can be constructed by integrating from $-R$ to R and then around the semicircle $Re^{i\theta}$ for $0 \leq \theta \leq \pi$ in the upper half plane. As $R \to \infty$ the integrand will become small on the semicircle due to the presence of the $e^{i\alpha x}$ term. Hence the integral along the real line will be given in terms of the residue at $\alpha = 2i$,

$$f(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha x} d\alpha}{(\alpha - 2i)(\alpha + i)} = -\frac{2\pi i}{2\pi} \left(\frac{e^{-2x}}{3i} \right) = -\frac{e^{-2x}}{3}.$$

If $x < 0$, the integral can be closed in the lower half plane, and thus will be given by the residue at $\alpha = -i$, plus an additional minus sign due to the contour being in the reverse direction.

$$f(x) = -\frac{-2\pi i}{2\pi} \left(-\frac{e^{x}}{3i} \right) = -\frac{e^{x}}{3}.$$

10. The first Laplace transform can be written as

$$F_1(p) = \frac{p + b}{(p + b + ia)(p + b - ia)}$$

and hence the Bromwich inversion integral

$$f_1(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(p) e^{pt} dp = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(p+b) e^{pt} dp}{(p + b + ia)(p + b - ia)}$$

has two simple poles at $p = -b \pm ia$. To evaluate this integral, the contour can be closed with a large semicircle in the left half plane, which will enclose both of the poles. The residues are

$$\operatorname{Res}\left(F_1(p) e^{pt}, p = -b \pm ia\right) = \lim_{p \to -b \pm ia} \frac{(p+b) e^{pt}}{p + b \pm ia} = \frac{iae^{(-b \pm ia)t}}{2ia} = \frac{e^{-bt \pm iat}}{2}.$$

Hence

$$f_1(t) = \frac{2\pi i}{2\pi i} \left(\frac{e^{-bt+iat}}{2} + \frac{e^{-bt-iat}}{2} \right) = e^{-bt} \frac{e^{iat} + e^{-iat}}{2} = e^{-bt} \cos at.$$

A plot of the function is shown in Fig. 5(a) for the case of $a = \pi$ and $b = 1/2$. The second Laplace transform can be written as

$$F_2(p) = \frac{2ap}{(p + ia)^2 (p - ia)^2}$$

and there Bromwich inversion integral

$$f_2(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2ape^{pt} dp}{(p + ia)^2 (p - ia)^2}$$

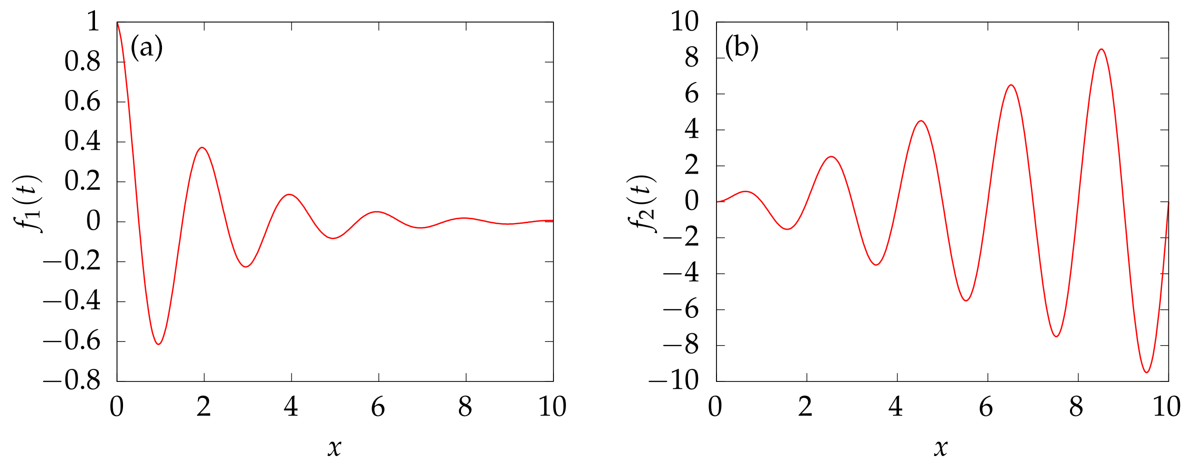

Figure 5: Plots of the two functions (a) $f_1(t) = e^{-bt}\cos at$ and (b) $f_2(t) = t\sin at$ considered in question 2, for the case of $a = \pi$ and $b = 1$.

has two poles of order 2 at $p = \pm ia$. Again, the contour can be closed in the left half plane, enclosing both poles. The residues are

$$\begin{aligned}
\text{Res}\left(F_2(p)e^{pt}, p = \pm ia\right) &= \lim_{p\to\pm ia} \frac{d}{dp}\frac{2ape^{pt}}{(p\pm ia)^2} \\
&= \lim_{p\to\pm ia} \frac{(p\pm ia)^2(2ae^{pt} + 2apte^{pt}) - (2ape^{pt})2(p\pm ia)}{(p\pm ia)^4} \\
&= \frac{-a^2(8ae^{\pm iat} \pm 8ia^2te^{\pm iat}) - (2iae^{\pm iat})2(2ia)}{16a^4} \\
&= \frac{(-8a^3 \mp 8ia^4t + 8a^3)e^{\pm iat}}{16a^4} = \frac{\pm te^{\pm iat}}{2i}.
\end{aligned}$$

Hence

$$f_2(t) = \frac{2\pi i}{2\pi i}\left(\frac{te^{iat}}{2i} - \frac{te^{-iat}}{2i}\right) = t\sin at$$

and the function is plotted in Fig. 5(b) for the case of $a = \pi$.

11. The integral can be written as

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x+i)}dx = \int_{-\infty}^{\infty} \frac{e^{iz}+e^{-iz}}{2(z+i)^2(z-i)}dz.$$

The integrand has a simple pole at $z = i$ and a pole of order 2 at $z = -i$. The term e^{ix} will become small in the upper half plane, and the term e^{-ix} will become small in the lower half plane. To make use of residue calculus, the integral must be split into two components

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{iz}}{2(z+i)^2(z-i)} dz, \qquad I_2 = \int_{-\infty}^{\infty} \frac{e^{-iz}}{2(z+i)^2(z-i)} dz,$$

which must be considered separately. For I_1 a closed contour can be made by using a large semicircle in the upper half plane, which will enclose the simple pole at $z = i$. The residue at this pole is

$$\text{Res}\left(\frac{e^{iz}}{2(z+i)^2(z-i)}, z = i\right) = \lim_{z \to i} \frac{e^{iz}}{2(z+i)^2} = -\frac{1}{8e}$$

and hence

$$I_1 = 2\pi i \frac{-1}{8e} = \frac{-i\pi}{4e}.$$

For I_2 a closed contour can be made by using a large semicircle in the lower half plane, which will enclose the pole at $z = -i$. The residue at this pole is

$$\begin{aligned}
\text{Res}\left(\frac{e^{-iz}}{2(z+i)^2(z-i)}, z = -i\right) &= \lim_{z \to -i} \frac{d}{dz}\left(\frac{e^{-iz}}{2(z-i)}\right) \\
&= \lim_{z \to -i}\left(\frac{-(z-i)ie^{-iz} - e^{-iz}}{2(z-i)^2}\right) \\
&= \left(\frac{2i^2e^{-1} - e^{-1}}{2(-2i)^2}\right) = \frac{3}{8e}
\end{aligned}$$

and hence

$$I_2 = -2\pi i \frac{3}{8e} = -\frac{3i\pi}{4e}$$

where an additional minus sign has been incorporated due to the contour being clockwise. Hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x+i)} dx = I_1 + I_2 = \frac{-i\pi}{e}.$$

12. (a) First note that since f is periodic with period 2π,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} f(x+z)dx$$

for any constant z, since the shifted integral will still cover the entire range of f. Let $f_s(x)$ have complex Fourier series $\sum_{-\infty}^{\infty} c_n^s e^{inx}$. Then

$$\begin{aligned}
c_0^s &= \frac{1}{4\pi l}\int_{-\pi}^{\pi} dx \int_{x-l}^{x+l} f(y)dy \\
&= \frac{1}{4\pi l}\int_{-\pi}^{\pi} dx \int_{-l}^{l} f(x+z)dz \\
&= \frac{1}{2l}\int_{-l}^{l} dz \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+z)dx\right) = \frac{1}{2l}\int_{-l}^{l} c_0 dz = c_0.
\end{aligned}$$

Similarly for $n \neq 0$,

$$\begin{aligned}
c_n^s &= \frac{1}{4\pi l}\int_{-\pi}^{\pi} e^{-inx}dx \int_{x-l}^{x+l} f(y)dy \\
&= \frac{1}{4\pi l}\int_{-\pi}^{\pi} e^{-inx}dx \int_{-l}^{l} f(x+z)dz \\
&= \frac{1}{2l}\int_{-l}^{l} e^{inz}dz \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in(x+z)} f(x+z)dx\right) \\
&= \frac{1}{2l}\int_{-l}^{l} e^{inz}c_n dz \\
&= \frac{c_n}{2l}\left[\frac{e^{inz}}{in}\right]_{-l}^{l} = \frac{c_n(e^{inl}-e^{-inl})}{2lin} = \frac{c_n \sin nl}{nl}.
\end{aligned}$$

If the sinc function is defined as

$$\operatorname{sinc} x = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0 \end{cases}$$

then it can be seen that in general $c_n^s = c_n \operatorname{sinc} nl$.

(b) The a_n coefficients of the Fourier series can be expressed in terms of the complex Fourier coefficients as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{inx}+e^{-inx}\right) f(x)dx = c_n + c_{-n}.$$

Similarly, the b_n coefficients can be expressed as

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx f(x)dx = \frac{1}{2\pi i}\int_{-\pi}^{\pi}\left(e^{inx}-e^{-inx}\right) f(x)dx = \frac{c_n - c_{-n}}{i}.$$

Hence, by converting to the complex Fourier series coefficients and back again, it can be seen that the Fourier coefficients of the smoothed series are

$$a_n^s = c_n^s + c_{-n}^s = c_n \operatorname{sinc} nl + c_{-n}\operatorname{sinc}(-nl) = (c_n + c_{-n})\operatorname{sinc} nl = a_n \operatorname{sinc} nl$$

and

$$b_n^s = \frac{c_n^s - c_{-n}^s}{i} = \frac{c_n \operatorname{sinc} nl - c_{-n}\operatorname{sinc}(-nl)}{i} = \frac{(c_n - c_{-n})\operatorname{sinc} nl}{i} = b_n \operatorname{sinc} nl.$$

13. (a) For the first function,

$$\frac{\partial u}{\partial x} = 3x^2 + y^2, \qquad \frac{\partial v}{\partial y} = -x^2 - 3y^2$$

and since $u_x \neq v_y$ the Cauchy–Riemann equations are not satisfied and this function is not analytic. For the second function

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

and the first Cauchy–Riemann equation $u_x = v_y$ is satisfied. In addition,

$$\frac{\partial u}{\partial y} = -6yx, \qquad \frac{\partial v}{\partial x} = 6yx$$

and thus the second Cauchy–Riemann equation $u_y = -v_x$ is satisfied also, and hence the function is analytic. (It can be verified that this function is equal to z^3.)

(b) The partial derivatives of the components of f are

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^{x^2-y^2}(2x\cos 2xy - 2y\sin 2xy), \\
\frac{\partial v}{\partial y} &= e^{x^2-y^2}(-2y\sin 2xy + 2x\cos 2xy), \\
\frac{\partial u}{\partial y} &= e^{x^2-y^2}(-2y\cos 2xy - 2x\sin 2xy), \\
\frac{\partial v}{\partial x} &= e^{x^2-y^2}(2x\sin 2xy + 2y\cos 2xy)
\end{aligned}$$

and it can be seen that $u_x = v_y$ and $u_y = -v_x$, so the Cauchy–Riemann equations are satisfied and hence f is analytic. To determine f as a function of z, in can be seen[1] that

$$f(x,y) = e^{x^2-y^2}e^{2ixy} = e^{x^2+2ixy-y^2} = e^{(x+iy)^2} = e^{z^2}.$$

14. The total cost of the tunnel is

$$\int_{-a}^{a} F(x,y,y')dx = \int_{-a}^{a} \sqrt{1+y}ds = \int_{-a}^{a} \sqrt{1+y}\sqrt{1+y'^2}dx.$$

Since the integrand has no explicit x dependence, the Beltrami identity can be used, and

$$C = F - y'\frac{\partial F}{\partial y'} = \sqrt{1+y}\sqrt{1+y'^2} - \frac{y'^2\sqrt{1+y}}{\sqrt{1+y'^2}} = \frac{\sqrt{1+y}}{\sqrt{1+y'^2}}$$

for some constant C. Hence

$$C^2 + C^2y'^2 = 1 + y$$

[1] In general, if there was no straightforward way to convert x and y into z, then the direct substitutions $x = (z+\bar{z})/2, y = (z-\bar{z})/2i$ could be employed. If f is analytic, the terms involving $\bar{z}$ should cancel.

and therefore

$$Cy' = \sqrt{1 - C^2 + y}.$$

Hence

$$\int \frac{C dy}{\sqrt{1 - C^2 + y}} = \int dx,$$

whereupon integration gives

$$2C\sqrt{1 - C^2 + y} = x - x_0$$

for some constant x_0. This can be rearranged to give

$$1 - C^2 + y = \frac{(x - x_0)^2}{4C^2}$$

and hence

$$y = C^2 - 1 + \frac{(x - x_0)^2}{4C^2}.$$

Now consider the boundary conditions, $y(\pm a) = 0$. By symmetry, $x_0 = 0$. To find C, note that

$$0 = y(a) = C^2 - 1 + \frac{a^2}{4C^2} = \frac{4C^4 - 4C^2 + a^2}{4C^2} = \frac{(2C^2 - 1)^2 - (1 - a^2)}{4C^2}$$

and therefore

$$C^2 = \frac{1 \pm \sqrt{1 - a^2}}{2},$$

giving two valid solutions for each value of a. The corresponding functional form is

$$y(x) = (C^2 - 1) + \frac{x^2}{4C^2} = \frac{x^2 - a^2}{4C^2} = \frac{x^2 - a^2}{2 \pm 2\sqrt{1 - a^2}}.$$

15. The two triangles together make up the square, $0 \le x \le 3$ and $0 \le y \le 3$. First consider any extrema in the interior of the square; they must satisfy

$$0 = \frac{\partial f}{\partial x} = 2x - y - 3, \qquad 0 = \frac{\partial f}{\partial y} = -x + 4y - 2.$$

From the first equation $y = 2x - 3$. Substituting into the second equation gives

$$0 = -x + 4(2x - 3) - 2 == 7x - 14$$

and therefore $x = 2$, from which it follows that $y = 1$. At this point, $f(x, y) = 4 - 2 + 2 - 6 - 2 = -4$. The boundaries could also contain possible extrema:

$$f(0, y) = 2y^2 - 2y, \qquad 0 = f_y(0, y) = 4y - 2, \qquad y = 1/2;$$

$$f(x,0) = x^2 - 3x, \qquad 0 = f_x(x,0) = 2x - 3, \qquad x = 3/2;$$
$$f(3,y) = -5y + 2y^2, \qquad 0 = f_y(3,y) = -5 + 4y, \qquad y = 5/4;$$
$$f(x,3) = x^2 - 6x + 12, \qquad 0 = f_x(x,3) = 2x - 6, \qquad x = 3.$$

The values of the function at these locations are

$$f(0, 1/2) = -1/2, \qquad f(3/2, 0) = -9/4, \qquad f(3, 5/4) = -25/8, \qquad f(3,3) = 3.$$

Note that one of these points coincides with a corner. The central line $x = y$ could also hold minima and maxima for either T_1 or T_2:

$$0 = \frac{d}{dx}(f(x,x)) = \frac{d}{dx}\left(2x^2 - 5x\right) = 4x - 5, \qquad x = 5/4.$$

In addition to this point, the three remaining corners could also be extrema, and function values at these points are

$$f(5/4, 5/4) = -25/8, \qquad f(0,0) = 0, \qquad f(3,0) = 0, \qquad f(0,3) = 12.$$

A contour plot of f is shown in Fig. 6. By looking at the possibilities, it can be seen that the minimum and maximum of f in T_1 are at $f(2,1) = -4$ and $f(3,3) = 3$ respectively. The minimum and maximum of f in T_2 are at $f(5/4, 5/4) = -25/8$ and $f(0,3) = 12$ respectively.

16. The integral

$$\oint_C \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

has a pole of order $n+1$ at $z = 0$. To evaluate the integral, the residue of the integral at $z =$ must be found. By using the binomial theorem,

$$\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} = \frac{1}{z}\sum_{k=0}^{2n} \binom{2n}{k} z^k \left(\frac{1}{z}\right)^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k} z^{2k-2n-1}.$$

The constant term corresponding to $1/z$ in the Laurent expansion will occur when $2k - 2n - 1 = -1$, which will be when $k = n$. The coefficient of this term is

$$\binom{2n}{n} = \frac{(2n!)}{n!(2n-n)!} = \frac{(2n)!}{(n!)^2}$$

and hence

$$\oint_C \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = 2\pi i \left(\frac{(2n)!}{(n!)^2}\right).$$

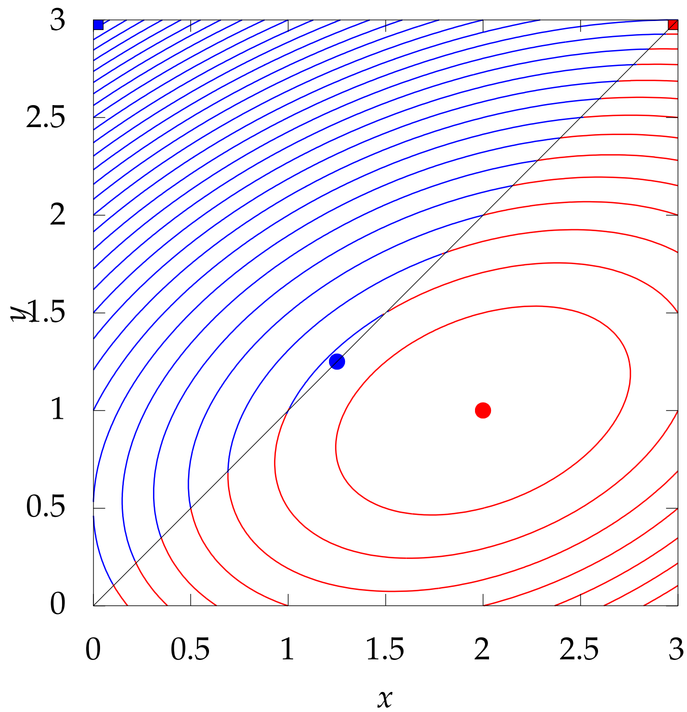

Figure 6: Contour plot of the function $f(x,y) = x^2 - xy + 2y^2 - 3x - 2y$, showing the region T_1 in red and the region T_2 in blue. For each region, the minimum is shown by a circle and the maximum is shown by a square. The contours are at intervals of $1/2$.

By using $\cos t = \frac{1}{2}(e^{it} + e^{-it})$ and the substitution $z = e^{it}$, so that $dz = ie^{it}dt$, the integral can evaluated as

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos^{2n} t \, dt &= \int_{-\pi}^{\pi} \left(\frac{e^{it} + e^{-it}}{2} \right)^{2n} dt \\
&= \frac{1}{2^{2n}} \int_{-\pi}^{\pi} \left(z + \frac{1}{z} \right)^{2n} \frac{dz}{iz} \\
&= \frac{1}{2^{2n} i} 2\pi i \left(\frac{(2n)!}{(n!)^2} \right) = \frac{\pi (2n)!}{2^{2n-1} (n!)^2}.
\end{aligned}$$

Solutions to the additional practice questions

17. The area and perimeter of the pentagon are given by

$$A = 2wh + wl, \qquad P = 2h + 2w + 2\sqrt{w^2 + l^2}$$

respectively. To maximize the area subject to a fixed perimeter, the augmented function

$$A(w,h,l,\lambda) = 2wh + wl + \lambda(P - 2h - 2w - 2\sqrt{w^2 + l^2})$$

can be considered where λ is a Lagrange multiplier. An extremal point therefore satisfies

$$\frac{\partial A}{\partial w} = 2h + l - 2\lambda - \frac{2w\lambda}{\sqrt{w^2 + l^2}} = 0 \quad (7)$$

$$\frac{\partial A}{\partial h} = 2w - 2\lambda = 0 \quad (8)$$

$$\frac{\partial A}{\partial l} = w - \frac{2l\lambda}{\sqrt{w^2 + l^2}} = 0. \quad (9)$$

From Eq. 8 it follows that $\lambda = w$. Equation 6 then becomes

$$w = \frac{2lw}{\sqrt{w^2 + l^2}}$$

and hence

$$\sqrt{w^2 + l^2} = 2l.$$

Therefore $w^2 = 3l^2$ and $w = l\sqrt{3}$. Substituting into Eq. 7 then gives

$$2h + l - 2l\sqrt{3} - \frac{6l^2}{\sqrt{4l^2}} = 0$$

and hence

$$h = (\sqrt{3} + 1)l.$$

Substituting into the constraint equation gives

$$P = 2(\sqrt{3} + 1)l + 2\sqrt{3}l + 2\sqrt{3l^2 + l^2} = (6 + 4\sqrt{3})l$$

and hence

$$l = \frac{P}{6 + 4\sqrt{3}} = \frac{(2\sqrt{3} - 3)P}{6}.$$

It follows that

$$h = \frac{(\sqrt{3} + 1)(2\sqrt{3} - 3)P}{6} = \frac{(3 - \sqrt{3})P}{6}$$

and

$$w = \frac{\sqrt{3}(2\sqrt{3} - 3)P}{6} = \frac{(6 - 3\sqrt{3})P}{6} = \frac{(2 - \sqrt{3})P}{2}.$$

The corresponding area is

$$A = w(2h + l) = \frac{(2 - \sqrt{3})(6 - 2\sqrt{3} + 2\sqrt{3} - 3)P^2}{12} = \frac{(2 - \sqrt{3})P^2}{4}.$$

The area evaluates to $A \approx 0.066987P^2$. Note that the area enclosed by a square of perimeter P is

$$A_s = \left(\frac{P}{4}\right)^2 = \frac{P^2}{16} = 0.0625P^2.$$

As would be expected, due to the extra degree of freedom provided by l, which creates a slightly rounder shape, the pentagon encloses slightly more area.

18. Plots of f and g are shown Fig. 7(a) and Fig. 7(b) respectively. The convolution is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

and since $g(y) = 1$ for $|y| < 1$ and zero otherwise, this can be written as

$$(f * g)(x) = \int_{-1}^{1} f(x-y)dy. \tag{10}$$

The function f will be non-zero for the range $-1 < x - y < 1$ which is equivalent to $x - 1 < y < x + 1$. If $x > 2$, then $x - 1 > 1$ and the integration will evaluate to zero. If $0 \leq x \leq 2$, then the integration will be non-zero over the range from $x - 1$ to 1, and thus

$$\begin{aligned} (f * g)(x) &= \int_{x-1}^{1} (1 - (y-x)^2)dy \qquad (11) \\ &= \frac{1}{3}\left[3y - (y-x)^3\right]_{x-1}^{1} \\ &= \frac{3 - (1-x)^3 - 3(x-1) + (x-1-x)^3}{3} \\ &= \frac{3 - 1 + 3x - 3x^2 + x^3 - 3x + 3 - 1}{3} \\ &= \frac{x^3 - 3x^2 + 4}{3} \\ &= \frac{(x+1)(x-2)^2}{3}. \end{aligned}$$

Note that both f and g are even. If $x < 0$, then $x = -z$ for $z > 0$, and

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{\infty} f(-z-y)g(y)dy \\ &= \int_{-\infty}^{\infty} f(-z+q)g(-q)dq \\ &= \int_{-\infty}^{\infty} f(z-q)g(q)dq \\ &= (f * g)(z) = (f * g)(-x) \end{aligned}$$

and hence $f * g$ is also even. Therefore

$$(f*g)(x) = \begin{cases} \frac{(x+1)(x-2)^2}{3} & \text{for } 0 \le x < 2, \\ \frac{(1-x)(x+2)^2}{3} & \text{for } -2 \le x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(x+1)(x-2)^2$ will equal zero when $x = 2$, the convolution is continuous there; by symmetry this will also be true at $x = -2$. The function $f * g$ is plotted in Fig. 7(c).

19. For the alternative form of g, where $g(x) = x$ for $-1 < x < 1$, the convolution will be described by Eq. 11 but with an additional factor of y:

$$\begin{aligned} (f*g)(x) &= \int_{x-1}^{1} y(1-(y-x)^2)dy \\ &= \int_{x-1}^{1} (y-(y-x)^3 - x(y-x)^2)dy \\ &= \frac{1}{12}\left[6y^2 - 3(y-x)^4 - 4x(y-x)^3\right]_{x-1}^{1} \\ &= \frac{6-3(1-x)^4 - 4x(1-x)^3 - 6(x-1)^2 + 3 - 4x}{12} \\ &= \frac{1}{12}(6 - 3 + 12x - 18x^2 + 12x^3 - 3x^4 - 4x \\ &\quad +12x^2 - 12x^3 + 4x^4 - 6x^2 + 12x - 6 + 3 - 4x) \\ &= \frac{x^4 - 12x^2 + 16x}{12} = \frac{x(x+4)(x-2)^2}{12}. \end{aligned}$$

For this case, f is even and g is odd. If $x < 0$, then $x = -z$ for $z > 0$, and

$$\begin{aligned} (f*g)(x) &= \int_{-\infty}^{\infty} f(-z-y)g(y)dy \\ &= \int_{-\infty}^{\infty} f(-z+q)g(-q)dq \\ &= -\int_{-\infty}^{\infty} f(z-q)g(q)dq \\ &= -(f*g)(z) = (f*g)(-x) \end{aligned}$$

and thus $f * g$ is odd. Plots of f, g, and $f * g$ are shown in Fig. 8.

It is worth noting that the integral in Eq. 10 can be approximated as a sum,

$$(f*g)(x) = \int_{-1}^{1} f(x-y)dy \approx \frac{1}{N}\sum_{n=0}^{N} f\left(x - \frac{2n-N}{N}\right).$$

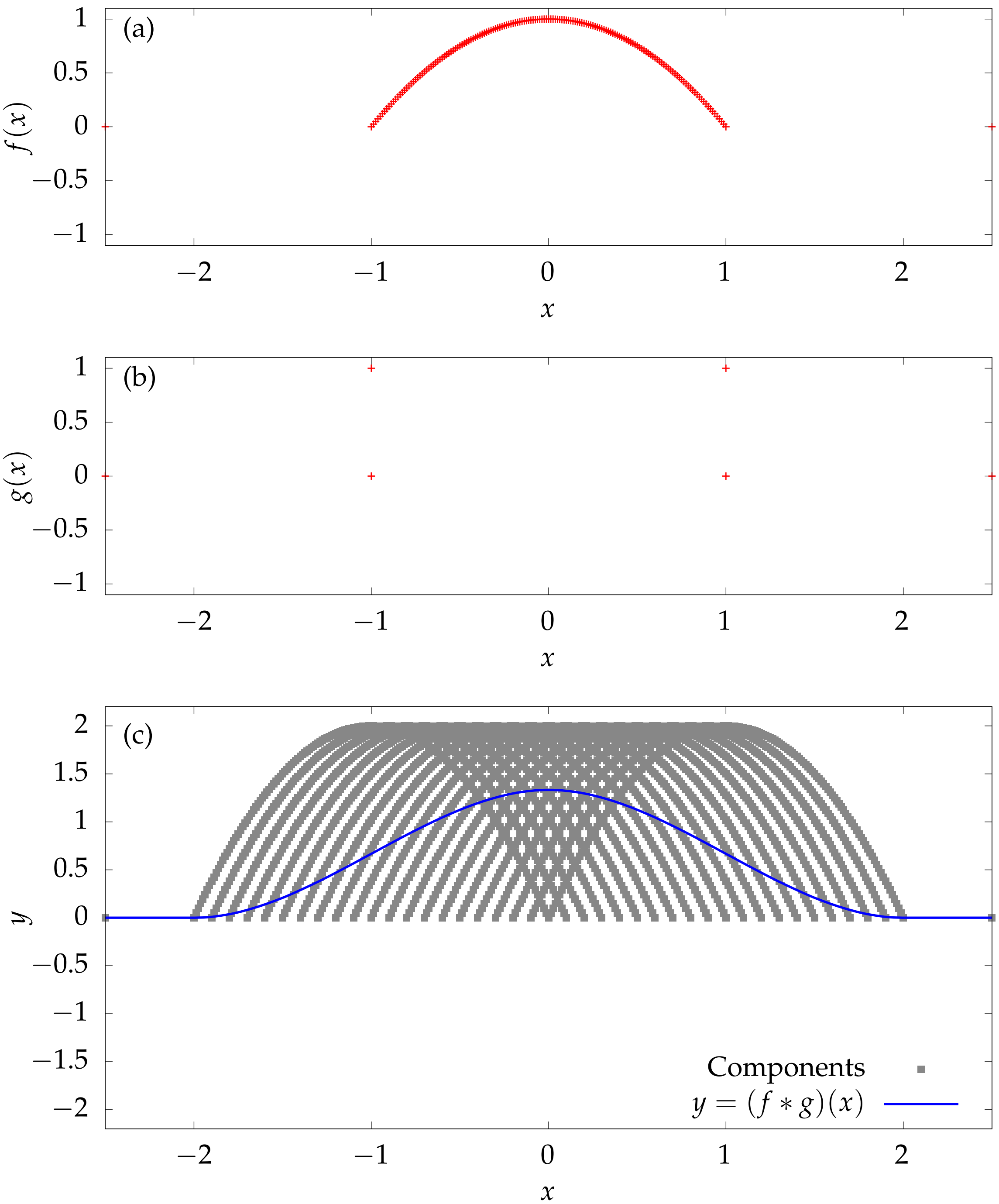

Figure 7: Two functions (a) f and (b) g, and (c) their convolution $f * g$.

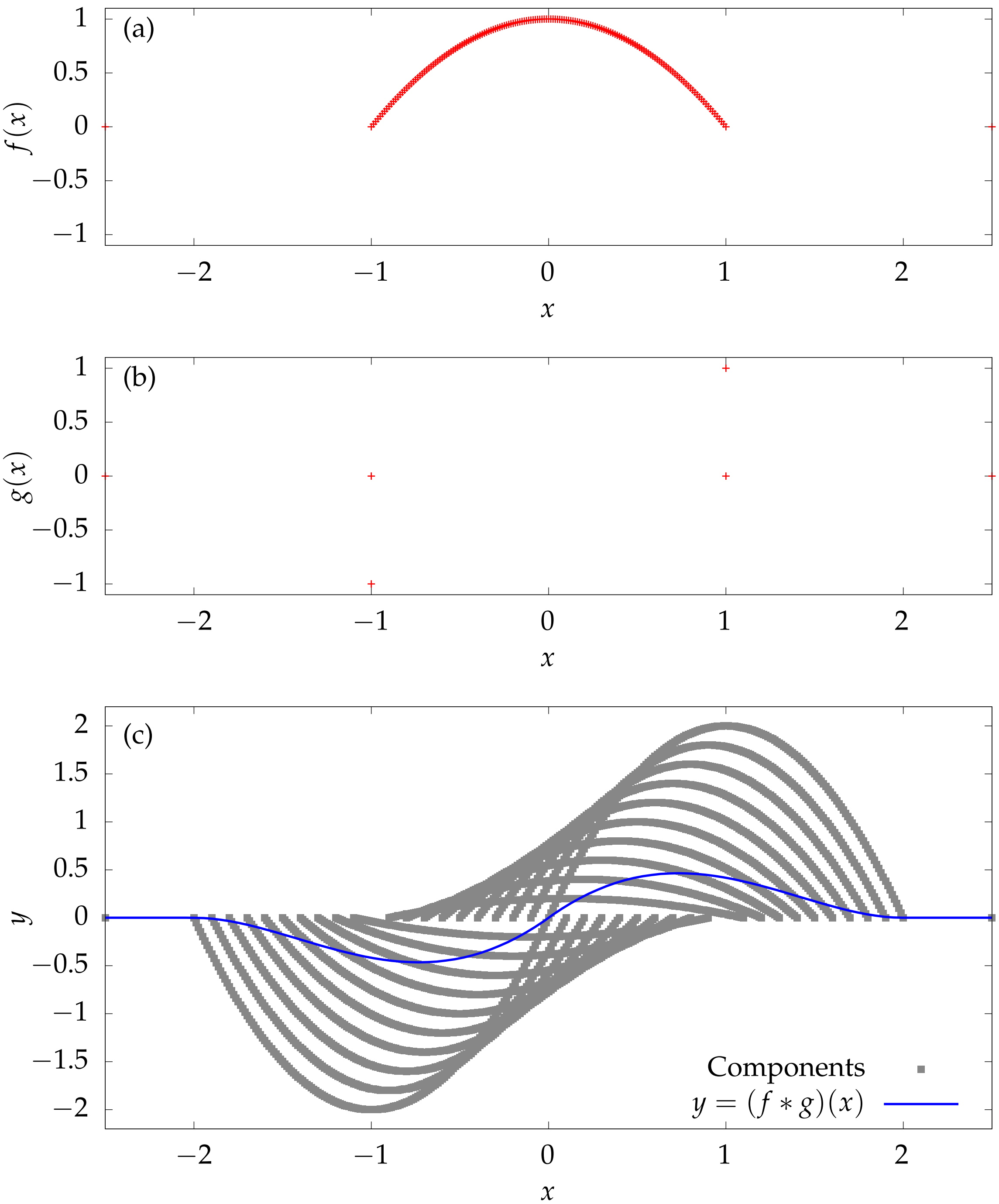

Figure 8: Two functions (a) f and (b) g, and (c) their convolution $f * g$.

Each of the terms $f(x - \frac{2n-N}{N})$ corresponds to a copy of f translated by $\frac{2n-N}{N}$. Hence the convolution can be viewed as a sum over a number of translated components. These are plotted in Fig. 7(c) and it can be seen the convolution is approximately an average of these components. For the alternative form of g, the convolution can be approximated as

$$(f * g)(x) = \int_{-1}^{1} y f(x-y) dy \approx \frac{1}{N} \sum_{n=0}^{N} f\left(x - \frac{2n-N}{N}\right) \frac{2n-N}{N}.$$

This case can be viewed similarly, although here the components are now weighted by the x factor. These components are plotted in Fig. 8(c).

20. Differentiating both sides of the geometric series formula gives

$$\sum_{n=0}^{\infty} n a^{n-1} = \frac{d}{da}\left(\frac{1}{1-a}\right) = \frac{1}{(1-a)^2}$$

and multiplying both sides by a gives

$$\sum_{n=0}^{\infty} n a^{n} = \frac{a}{(1-a)^2}.$$

Hence

$$f(x) = \frac{\lambda e^{ix}}{(1-\lambda e^{ix})^2} = \sum_{n=0}^{\infty} n \lambda^n e^{inx}.$$

This is an expression for f as a complex Fourier series with coefficients

$$c_n = \begin{cases} n\lambda^n & \text{for } n > 0, \\ 0 & \text{for } n \leq 0. \end{cases}$$

The smoothed Fourier series is therefore given by

$$\begin{aligned}
f_s(x) &= \sum_{n=1}^{\infty} \frac{n\lambda^n \sin nl}{nl} e^{inx} \\
&= \frac{1}{2il} \sum_{n=1}^{\infty} \lambda^n (e^{inl} - e^{-inl}) e^{inx} \\
&= \frac{1}{2il} \sum_{n=1}^{\infty} \lambda^n (e^{in(x+l)} - e^{in(x-l)}) \\
&= \frac{1}{2il} \left(\frac{1}{1-\lambda e^{i(x+l)}} - \frac{1}{1-\lambda e^{i(x-l)}} \right) \\
&= \frac{1}{2il} \left(\frac{1}{1-\lambda e^{i(x+l)}} - \frac{1}{1-\lambda e^{i(x-l)}} \right) \\
&= \frac{\lambda(e^{i(x+l)} - e^{i(x-l)})}{2il(1-\lambda e^{i(x+l)})(1-\lambda e^{i(x-l)})} \\
&= \frac{\lambda e^{ix} \sin l}{l(1 - 2\lambda e^{ix} \cos l + \lambda^2 e^{2ix})}.
\end{aligned}$$

It can be seen that if $l \to 0$, then $\frac{\sin l}{l} \to 1$ and $\cos l \to 1$, and the function will tend to f as expected.

21. (a) The radial distance can be calculated as

$$\begin{aligned}
r &= \sqrt{a^2b^2 + \frac{(a^2-b^2)^2}{4}} = \frac{\sqrt{4a^2b^2 + a^4 - 2a^2b^2 + b^4}}{2} \\
&= \frac{\sqrt{(a^2+b^2)^2}}{2} = \frac{a^2+b^2}{2}.
\end{aligned}$$

(b) First note that $\dot{x} = \dot{a}b + a\dot{b}$ and $\dot{y} = \dot{a}a - \dot{b}b$, and hence

$$\begin{aligned}
L &= \frac{m(\dot{x}^2+\dot{y}^2)}{2} - \left(-\frac{m}{r}\right) = \frac{m((\dot{a}b + a\dot{b})^2 + (\dot{a}a - \dot{b}b)^2)}{2} + \frac{2m}{a^2+b^2} \\
&= \frac{m(\dot{a}^2b^2 + a^2\dot{b}^2 + \dot{a}^2a^2 + \dot{b}^2b^2)}{2} + \frac{2m}{a^2+b^2} \\
&= \frac{m(\dot{a}^2+\dot{b}^2)(a^2+b^2)}{2} + \frac{2m}{a^2+b^2}.
\end{aligned}$$

(c) The Euler–Lagrange equation for $b(t)$ can be calculated as

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}}\right) - \frac{\partial L}{\partial b} = \frac{d}{dt}\left(m(a^2+b^2)\dot{b}\right) - m(\dot{a}^2+\dot{b}^2)b + \frac{4mb}{(a^2+b^2)^2}$$

and hence

$$0 = 2(a\dot{a} + b\dot{b})\dot{b} + (a^2 + b^2)\ddot{b} - (\dot{a}^2 + \dot{b}^2)b + \frac{4b}{(a^2 + b^2)^2},$$

which can be simplified to give

$$0 = \ddot{b}(a^2 + b^2) + \dot{b}(2a\dot{a} + b\dot{b}) - \dot{a}^2 b + \frac{4b}{(a^2 + b^2)^2}. \tag{12}$$

Since the Lagrangian is invariant if a and b are switched, it follows that the Euler–Lagrange equation for $a(t)$ is

$$0 = \ddot{a}(a^2 + b^2) + \dot{a}(2b\dot{b} + a\dot{a}) - \dot{b}^2 a + \frac{4a}{(a^2 + b^2)^2}. \tag{13}$$

If $a(t) = C$ where C is a constant, then Eq. 13 becomes

$$\dot{b}^2 = \frac{4}{(C^2 + b^2)^2} \tag{14}$$

and hence

$$\dot{b} = \frac{2}{C^2 + b^2}, \tag{15}$$

where the positive sign for the square root has been chosen. Differentiating gives

$$\ddot{b} = -\frac{4\dot{b}b}{C^2 + b^2} = -\frac{8b}{(C^2 + b^2)^3}. \tag{16}$$

For $a(t) = C$ to lead to a valid solution, Eq. 12 must also be satisfied. It can be seen that the equation becomes

$$\ddot{b}(C^2 + b^2) + b\dot{b}^2 = -\frac{4b}{(C^2 + b^2)^2} \tag{17}$$

and using Eqs. 14 and 16 it can be seen that the left hand side is

$$\begin{aligned}
\ddot{b}(C^2 + b^2) + b\dot{b}^2 &= \left(-\frac{8b}{(C^2 + b^2)^3}\right)(C^2 + b^2) + \frac{4b}{(C^2 + b^2)^2} \\
&= -\frac{8b}{(C^2 + b^2)^2} + \frac{4b}{(C^2 + b^2)^2} \\
&= -\frac{4b}{(C^2 + b^2)^2}.
\end{aligned}$$

Hence Eq. 17 is satisfied and $a(t) = C$ is a consistent solution. To determine the evolution of b, note that Eq. 15 can be rearranged to give

$$\int (C^2 + b^2)db = \int 2dt$$

and hence

$$C^2 b + \frac{b^3}{3} = 2(t - t_0)$$

for some constant t_0. Using the given boundary condition of $b(0) = 0$, it follows that $t_0 = 0$. This expression cannot easily be written as a function $b(t)$, since it would involve solving a cubic for b, but it is possible to write

$$t(b) = \frac{b(3C^2 + b^2)}{6}.$$

When $|b|$ is large it can be seen that $b(t) \approx \sqrt[3]{6t}$. Note that the trajectory $(a(t), b(t))$ is a parabola, which is physically reasonable for a mass moving in a gravitational field—many comets follow near-parabolic trajectories as they orbit the Sun.

A Appendix: Derivation of the Green function condition

In question 1, when solving the equation

$$y'' + k^2 y = \delta(x - x') \tag{18}$$

a solution of the form

$$y(x) = \begin{cases} A\cos kx + B\sin kx & \text{for } x < x', \\ C\cos kx + D\sin kx & \text{for } x > x' \end{cases}$$

is constructed, and after the boundary conditions $y(0) = y(\frac{\pi}{2k}) = 0$ are taken into account, this becomes

$$y(x) = \begin{cases} B\sin kx & \text{for } x < x', \\ C\cos kx & \text{for } x > x'. \end{cases}$$

To set the two remaining constants B and C, conditions to match the two sections of the solution at $x = x'$ are needed. Let the solutions for $x < x'$ and $x > x'$ be called $y_-(x)$ and $y_+(x)$ respectively. The solution y is taken to be continuous at $x = x'$ and thus $y_-(x') = y_+(x')$, which gives

$$B\sin kx' = C\cos kx' \tag{19}$$

To get another condition, the derivatives of y_- and y_+ need to be considered. Physically, if $y(x)$ is viewed as the position of a mass as a function of time, the delta function term in Eq. 18 corresponds to applying an impulsive force to the mass at the time $x = x'$. In the same way that a ball hitting a wall experiences a short impulsive force over a small interval, which causes a rapid change in its velocity, it should be expected that $\delta(x - x')$ will cause a change in the derivative of y at $x = x'$.

To analyze this mathematically, integration must be used. The total impulse of a delta function is given by

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

and since this is localized at $x = 0$, restricting to a small interval from $-\epsilon$ to ϵ gives

$$\int_{-\epsilon}^{\epsilon} \delta(x) = 1. \tag{20}$$

Now consider integrating Eq. 18 over the interval from $x' - \epsilon$ to $x' + \epsilon$. This gives

$$\int_{x'-\epsilon}^{x'+\epsilon} (y'' + k^2 y) dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x') dx.$$

The integral of y'' will be y', and the integral of the delta function is a shifted version of Eq. 20, and therefore

$$\left[y'\right]_{x'-\epsilon}^{x'+\epsilon} dx + \int_{x'-\epsilon}^{x'+\epsilon} k^2 y \, dx = 1.$$

As $\epsilon \to 0$, the integral of $k^2 y$ will tend to zero, since y is continuous and the integration range gets smaller. Since $y'(x' + \epsilon) \to y'_+(x')$ and $y'(x' - \epsilon) \to y'_-(x')$, it follows that

$$y'_+(x') - y'_-(x') = 1 \tag{21}$$

and therefore

$$(-Ck \sin kx') - Bk \cos kx' = 1. \tag{22}$$

Equations 19 and 22 can then be solved to find B and C, from which a complete solution to Eq. 18 can be constructed.

The condition in Eq. 21 is generally applicable. For any differential equation like Eq. 18 with the form

$$y'' + f(x)y' + g(x)y = \delta(x - x')$$

for some functions $f(x)$ and $g(x)$, the same derivation could be followed to obtain Eq. 21.