Math 121A: Sample midterm solutions

1. To find a Green function solution, consider solving the given equation for an impulsive input of the form $f(x) = \delta(x - x')$. In the regions $x > x'$ and $x < x'$, solutions of the form $y = e^{mx}$ can be searched for, which upon substitution into the differential equation gives
$$m^2 - k^2 = 0$$
and hence $m = \pm k$. To satisfy the boundary conditions as $x \to \pm\infty$, the solution must be of the form
$$y(x) = \begin{cases} Ae^{mx} & \text{for } x < x', \\ Be^{-mx} & \text{for } x > x', \end{cases}$$
where A and B are constants. To satisfy continuity at x' it follows that
$$Ae^{mx'} = Be^{-mx'}$$
and to satisfy $y'_+(x') - y'_-(x') = 1$ it follows that
$$-Bme^{-mx'} - Ame^{mx'} = 1$$
and thus
$$A = -\frac{e^{-mx'}}{2m}, \qquad B = -\frac{e^{mx'}}{2m}$$
so the solution can be written as
$$y(x) = -\frac{e^{-m|x-x'|}}{2m}.$$
Hence the Green function solution for an arbitrary source term $f(x)$ can be written as
$$y(x) = \int_{-\infty}^{\infty} \left(-\frac{e^{-m|x-x'|}}{2m} \right) f(x')dx'.$$

2. (a) The Fourier transform is
$$\begin{aligned}
\tilde{h}_\lambda(\alpha) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\lambda x^2 - ix\alpha\right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\lambda \left(x - \frac{i\alpha}{2\lambda}\right)^2 - \frac{\alpha^2}{4\lambda}\right) dx \\
&= \frac{e^{-\frac{\alpha^2}{4\lambda}}}{2\pi} \sqrt{\frac{\pi}{\lambda}} \\
&= \frac{e^{-\frac{\alpha^2}{4\lambda}}}{\sqrt{4\pi\lambda}}.
\end{aligned}$$

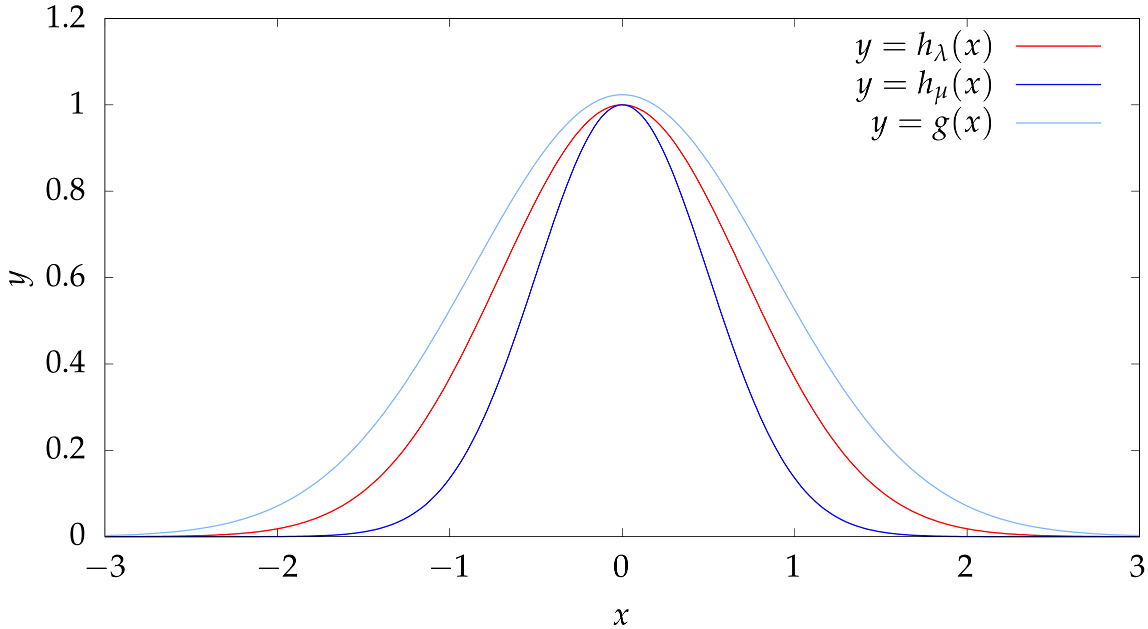

Figure 1: Graphs of the Gaussians considered in question 2, for the case of $\lambda = 1$ and $\mu = 2$.

(b) The convolution is given by

$$
\begin{aligned}
(h_\lambda * h_\mu)(x) &= \int_{-\infty}^{\infty} h_\lambda(\xi) h_\mu(x-\xi) d\xi \\
&= \int_{-\infty}^{\infty} \exp\left(-\lambda\xi^2 - \mu(x-\xi)^2\right) d\xi \\
&= \int_{-\infty}^{\infty} \exp\left(-(\lambda+\mu)\xi^2 + 2\mu x\xi - \mu x^2\right) d\xi \\
&= \int_{-\infty}^{\infty} \exp\left(-(\lambda+\mu)\left(\xi - \frac{\mu x}{(\lambda+\mu)}\right)^2 + \frac{\mu^2 x^2}{\lambda+\mu} - \mu x^2\right) d\xi \\
&= \sqrt{\frac{\pi}{\lambda+\mu}} \exp\left(+\frac{\mu^2 x^2}{\lambda+\mu} - \frac{\mu^2+\lambda\mu}{\lambda+\mu} x^2\right) \\
&= \sqrt{\frac{\pi}{\lambda+\mu}} \exp\left(-\frac{\lambda\mu}{\lambda+\mu} x^2\right).
\end{aligned}
$$

(c) If $\lambda = 1$ and $\mu = 2$ then

$$
g(x) = \sqrt{\frac{\pi}{3}} \exp\left(-\frac{2x^2}{3}\right).
$$

and thus g is a Gaussian that is wider than both h_λ and h_μ. The three curves are plotted in Fig. 1.

(d) Using the result from part (a) it can be seen that

$$
2\pi\tilde{h}_\lambda(\alpha)\tilde{h}_\mu(\alpha) = \frac{\exp\left(-\frac{\alpha^2}{4}\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)\right)}{2\sqrt{\lambda\mu}}.
$$

Using the result from part (b),

$$
\begin{aligned}
\tilde{g}(\alpha) &= \sqrt{\frac{\pi}{\lambda+\mu}} \frac{\exp\left(-\frac{\alpha^2}{4}\left(\frac{\lambda+\mu}{\lambda\mu}\right)\right)}{\sqrt{4\pi\lambda\mu}} \sqrt{\lambda+\mu} \\
&= \frac{\exp\left(-\frac{\alpha^2}{4}\left(\frac{1}{\lambda} + \frac{1}{\mu}\right)\right)}{2\sqrt{\lambda\mu}}
\end{aligned}
$$

and thus the two expressions are equal as expected.

3. (a) Taking the Fourier transform in x and the Laplace transform in t gives

$$
\tilde{F}(\alpha, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_0^{\infty} dt\, f(x,t) e^{-pt-ix\alpha}.
$$

Taking the transform of the equation $f_t + cf_x = bf_{xx}$ gives

$$p\tilde{F} - \tilde{f}(\alpha, 0) + ci\alpha\tilde{F} = b(i\alpha)^2\tilde{F}.$$

The Fourier transform of the initial condition is

$$\tilde{f}(\alpha, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(x)e^{-ix\alpha}dx = \frac{1}{2\pi}$$

and hence the transformed equation can be written as

$$(p + ic\alpha + b\alpha^2)\tilde{F} = \frac{1}{2\pi}.$$

Therefore

$$\tilde{F}(\alpha, p) = \frac{1}{2\pi(p + ic\alpha + b\alpha^2)}.$$

Since the Laplace transform of e^{-qt} is $1/(p+q)$, it follows that

$$\tilde{f}(\alpha, t) = \frac{e^{-(ic\alpha + b\alpha^2)t}}{2\pi}$$

Taking the inverse Fourier transform gives

$$\begin{aligned} f(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(ic\alpha+b\alpha^2)t + i\alpha x} d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-bt\left(\alpha - \frac{i(x-ct)}{2bt}\right)^2 - \frac{(x-ct)^2}{4bt}\right) d\alpha \\ &= \frac{1}{\sqrt{4\pi bt}} \exp\left(-\frac{(x-ct)^2}{4bt}\right). \end{aligned}$$

(b) By making use of part (a) and translational symmetry in x, it can be seen that if the initial condition is $g(x) = \delta(x - x')$ then the solution is

$$f(x,t) = \frac{1}{\sqrt{4\pi bt}} \exp\left(-\frac{(x - x' - ct)^2}{4bt}\right).$$

The Green function solution for an arbitrary initial condition $g(x)$ is therefore

$$f(x,t) = \frac{1}{\sqrt{4\pi bt}}\int_{-\infty}^{\infty} g(x') \exp\left(-\frac{(x - x' - ct)^2}{4bt}\right) dx'.$$

(c) For the case of $g(x) = e^{-ax}$ the solution is given by

$$\begin{aligned}
f(x,t) &= \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x'-ct)^2}{4bt} - ax'\right) dx' \\
&= \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} \exp\left(-\frac{x'^2 + 2(ct-x)x' + (ct-x)^2 + 4btax'}{4bt}\right) dx' \\
&= \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} \exp\left(-\frac{x'^2 + 2(ct-x+2bta)x' + (ct-x)^2}{4bt}\right) dx' \\
&= \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x'+ct-x+2bta)^2}{4bt}\right. \\
&\quad \left. + \frac{(ct-x+2bta)^2 - (ct-x)^2}{4bt}\right) dx' \\
&= \frac{1}{\sqrt{4\pi bt}} \sqrt{4bt\pi} \exp\left(\frac{(ct-x)4bta + 4b^2t^2a^2}{4bt}\right) \\
&= e^{-ax+ta(c+ba)}.
\end{aligned}$$

It can be seen that $f(x,0) = e^{-ax}$ and hence the initial condition is satisfied. The left hand side of the partial differential equation is

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = a(c+ba)f - caf = ba^2 f$$

and the right hand side is

$$b\frac{\partial^2 f}{\partial x^2} = ba^2 f$$

so the partial differential equation is satisfied.

4. (a) The zeroth cosine term is

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi}\pi = 1.$$

The other cosine terms are given by

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
&= \frac{1}{\pi} \int_{0}^{\pi} \cos nx \, dx \\
&= \frac{1}{\pi} \left[\frac{\sin nx}{n}\right]_0^{\pi} \\
&= 0.
\end{aligned}$$

The sine terms are given by

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx \\ &= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\ &= \frac{1}{n\pi}\left[1-\cos n\pi\right] \\ &= \begin{cases} \frac{2}{n\pi} & \text{for } n \text{ odd,} \\ 0 & \text{for } n \text{ even.} \end{cases} \end{aligned}$$

Hence

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2\sin nx}{n\pi}.$$

(b) The left hand side of the Parseval equation is

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{\pi}\int_{0}^{\pi} dx = 1$$

and the right hand side is

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

Equating the two gives

$$\frac{1}{2} = \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

which can be rearranged as

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

5. First note that the Fourier transform of $f(t) = e^{-at}$ is

$$F(p) = \int_0^{\infty} e^{-pt}\,dt = \int_0^{\infty} e^{-(a+p)t}\,dt = \left[\frac{e^{-(a+p)t}}{a+p}\right]_0^{\infty} = \frac{1}{a+p}.$$

The Laplace transform of the equation is therefore given by

$$p^2Y(p) - py(0) - y'(0) + 6(pY(p) - y(0)) + 8Y(p) = \frac{1}{p+3}$$

and hence

$$(p^2 + 6p + 8)Y(p) = (p + 6) + \frac{1}{p + 3}.$$

Therefore

$$\begin{aligned}
Y(p) &= \frac{p+6}{(p+2)(p+4)} + \frac{1}{(p+2)(p+3)(p+4)} \\
&= \frac{1}{p+2} + \frac{2}{(p+2)(p+4)} + \frac{1}{2(p+3)}\left(\frac{1}{p+2} - \frac{1}{p+4}\right) \\
&= \frac{1}{p+2} + \frac{1}{p+2} - \frac{1}{p+4} + \frac{1}{2}\left(\frac{1}{p+2} - \frac{2}{p+3} + \frac{1}{p+4}\right) \\
&= \frac{5}{2(p+2)} - \frac{1}{p+3} - \frac{1}{2(p+4)}
\end{aligned}$$

and hence

$$y(t) = \frac{5e^{-2t} - 2e^{-3t} - e^{-4t}}{2}.$$