Sample midterm 1 solutions

1. For the first series, note that
$$\frac{1}{4^n + (1/2)^n} < \frac{1}{4^n}$$
for all n, and
$$\sum_{n=0}^{\infty} \frac{1}{4^n}$$
is a convergent geometric series. Hence, by the comparison test,
$$\sum_{n=0}^{\infty} \frac{1}{4^n + (1/2)^n}$$
is a convergent series. For the second series, it can be seen that the terms $\sin n\pi/4$ do not converge to zero as n increases. Hence, by the preliminary test, it follows that the series does not converge. For the final series, consider the integral
$$\int \frac{1}{x \log x} dx = \log(\log(x)) + C.$$
Since $\log x \to \infty$ as $x \to \infty$, it follows that $\log(\log(x)) \to \infty$ as $x \to \infty$. Hence, by using the integral test, the series
$$\sum_{n=2} \frac{1}{n \log n}$$
diverges.

2. (a) If the coefficients in the power series are defined to be $a_n = 3^n$, then the radius of convergence can be found by first calculating
$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \right| = \frac{1}{3}.$$
Hence the radius of convergence is $R = 1/\rho = 3$. To determine the exact interval of convergence, consider the endpoints at $y = \pm R$. If $y = 3$, then
$$\sum_{n=0}^{\infty} \frac{y^n}{3^n} = \sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1,$$
which diverges by the preliminary test. If $y = -3$, then
$$\sum_{n=0}^{\infty} \frac{y^n}{3^n} = \sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n,$$
which also diverges by the preliminary test. Hence the exact interval of convergence is $-3 < y < 3$.

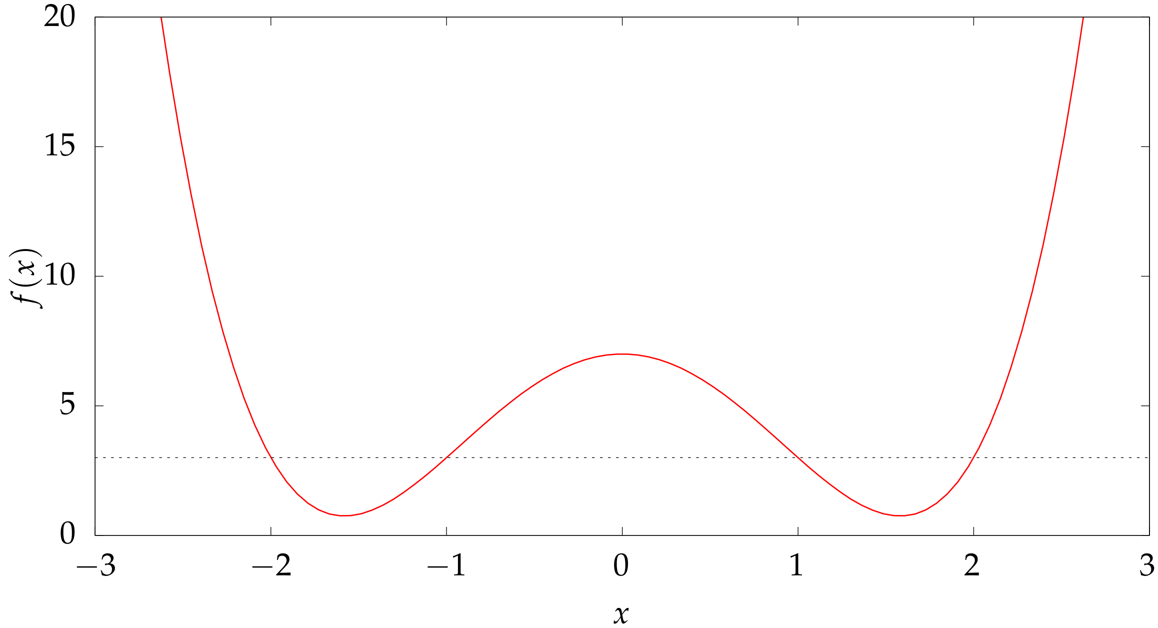

Figure 1: Sketch of the quartic function $f(x) = 3 + (x^2 - 1)(x^2 - 4)$ considered in question 2. The thin dashed line corresponds to $y = 3$.

(b) The function is plotted in Fig. 1. To sketch this function, note that it is even, and takes the value of 3 for $x = \pm 1, \pm 2$.

(c) The series

$$\sum_{n=0}^{\infty} \frac{f(x)^n}{3^n}$$

will converge if and only if $-3 < f(x) < 3$. By reference to Fig. 1, it can be seen that this will occur if $-2 < x < -1$ or $1 < x < 2$.

3. Let the triangle occupy the domain T. Its mass is given by

$$\begin{aligned} m = \int_T \rho(x,y)\, dx\, dy &= \int_0^1 \left(\int_0^{1-x} \rho(x,y) dy \right) dx \\ &= \int_0^1 \left(\int_0^{1-x} y dy \right) dx \\ &= \frac{1}{2} \int_0^1 (1 - 2x + x^2)\, dx \\ &= \frac{1}{2} \left(1 - \frac{2}{2} + \frac{1}{3} \right) = \frac{1}{6} \end{aligned}$$

To find the center of mass, the first moments in the x direction,

$$\begin{aligned}
\int_T x\rho(x,y)\,dx\,dy &= \int_0^1 x\left(\int_0^{1-x}\rho(x,y)dy\right)dx \\
&= \int_0^1 x\left(\int_0^{1-x} ydy\right)dx \\
&= \int_0^1 x\left(\frac{(1-x)^2}{2}\right)dx \\
&= \frac{1}{2}\int_0^1 (x-2x^2+x^3)dx \\
&= \frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right) = \frac{1}{24},
\end{aligned}$$

and in the y direction,

$$\begin{aligned}
\int_T y\rho(x,y)\,dx\,dy &= \int_0^1 \left(\int_0^{1-x} y\rho(x,y)dy\right)dx \\
&= \int_0^1 \left(\int_0^{1-x} y^2dy\right)dx \\
&= \int_0^1 \left(\frac{(1-x)^3}{3}\right)dx \\
&= \frac{1}{3}\int_0^1 (1-3x+3x^2-x^3)dx \\
&= \frac{1}{3}\left(1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}\right) = \frac{1}{12}
\end{aligned}$$

can be computed. The center of mass is then given by $6(1/24, 1/12) = (1/4, 1/2)$.

4. To begin, any extrema within the interior or the square can be found by taking partial derivatives,

$$\frac{\partial f}{\partial x} = 2(x-2y)-1, \qquad \frac{\partial f}{\partial y} = -4(x-2y)$$

and then searching for values of x and y where both expressions are zero. However, the second equation implies that $x = 2y$, and then the first equation is not satisfied. Hence there are no extrema in the interior of the square. On the horizontal boundaries, extrema can be found by differentiation,

$$f(x,-1) = (x+2)^2 + x = x^2+5x+4, \qquad 0 = f_x(x,-1) = 2x+5, \qquad x = -5/2,$$

$$f(x,1) = (x-2)^2 - x = x^2-5x+4, \qquad 0 = f_x(x,1) = 2x-5, \qquad x = +5/2,$$

but both values where the extrema are achieved lie outside the square. On the vertical boundaries, extrema are given by

$$f(-1,y) = (1+2y)^2 + 1 = 4y^2+4y+2, \qquad 0 = f_y(-1,y) = 8y+4, \qquad y = -1/2,$$

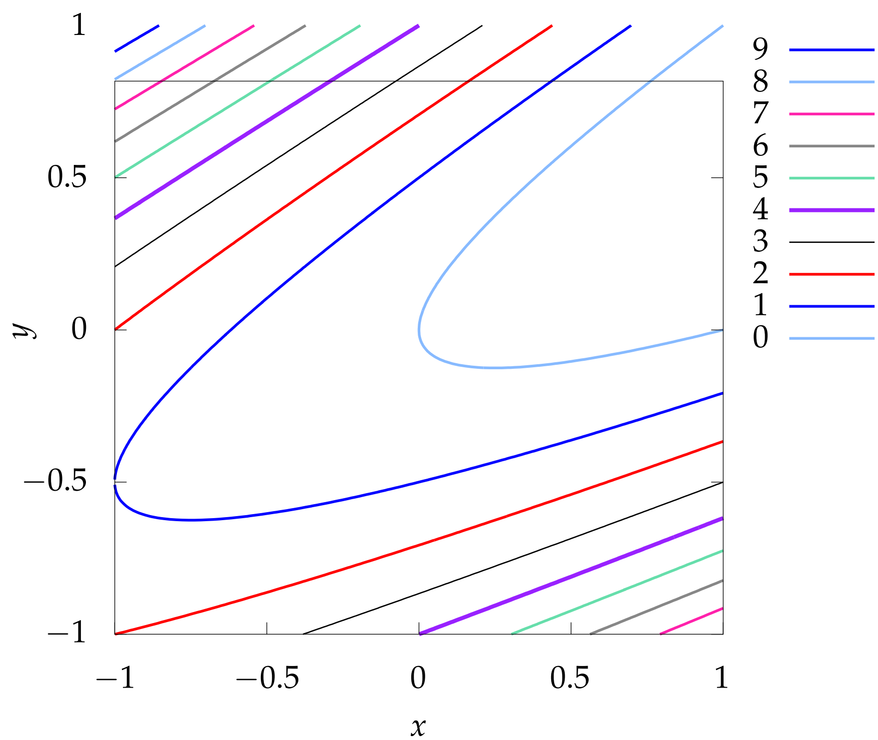

Figure 2: Contours of the function $f(x, y) = (x - 2y)^2 - x$ considered in question 4.

$$f(1,y) = (1-2y)^2 - 1 = 4y^2 - 4y, \qquad 0 = f_y(1,y) = 8y - 4, \qquad y = 1/2,$$

which give two values, $f(-1,-1/2) = 1$ and $f(1,1/2) = -1$. Finally, the extrema could be achieved at corners:

$$f(-1,-1) = 2, \qquad f(-1,1) = 10, \qquad f(1,-1) = 8, \qquad f(1,1) = 0.$$

Hence the minimum value is -1 attained at $(1,1/2)$ and the maximum value is 10 attained at $(-1,1)$. Contours of the function are shown in Fig. 2, which confirm these results.

5. The area of the tent surface is given by

$$A = lh + l\sqrt{h^2 + l^2}. \tag{1}$$

To maximize the volume while keeping the area constant, a Lagrange multiplier λ can be introduced, and the augmented function

$$V(h,l,\lambda) = \frac{hl^2}{2} + \lambda\left(lh + l\sqrt{h^2+l^2} - A\right).$$

can then be maximized. Taking partial derivatives with respect to the first three parameters gives

$$0 = \frac{\partial V}{\partial h} = \frac{l^2}{2} + \lambda\left(l + \frac{lh}{R}\right) \tag{2}$$

$$0 = \frac{\partial V}{\partial l} = hl + \lambda\left(h + R + \frac{l^2}{R}\right) \tag{3}$$

where $R = \sqrt{h^2 + l^2}$. For a physical solution, it can be assumed that $h > 0$ and $l > 0$. Equation 2 can be rearranged to give

$$\lambda = -\frac{Rl}{2(R+h)},$$

which can be substituted into Eq. 3 to give

$$hl = \frac{Rl}{2(R+h)}\left(h + R + \frac{l^2}{R}\right).$$

This can be rearranged as

$$2h(R+h) = Rh + R^2 + l^2$$

and hence

$$Rh = 2l^2 - h^2.$$

Squaring both sides gives

$$R^2h^2 = 4l^4 + h^4 - 4h^2l^2.$$

and since $R^2 = h^2 + l^2$ it follows that

$$h^4 + h^2l^2 = 4l^4 + h^4 - 4h^2l^2$$

so

$$5h^2 = 4l^2.$$

Therefore $h = 2l/\sqrt{5}$. Substituting into Eq. 1 gives

$$A = l^2\left(\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{5}}\right) = l^2\sqrt{5}.$$

and hence $l = 5^{-1/4}\sqrt{A}$ and $h = 2 \times 5^{-3/4}\sqrt{A}$. The volume is then given by

$$V = \frac{l^2h}{2} = 5^{-5/4}A^{3/2}.$$