

Math 121A: Final exam information

- The final exam will take place on Wednesday May 15th from 7pm–10pm in 180 Tan Hall.
- The exam is closed book – no textbooks, notebooks, or calculators allowed. As on the midterms, you will be expected to be familiar with the basic definitions, and know the key results.
- The final will consist of ten questions that will be of a similar style to those on the midterms. This will give you slightly longer per question than on the midterms.
- The exam will cover all parts of the course with equal weighting. In Boas, it will cover Chapters 1, 2, 3 (excluding §3.13), 4, 5, 7, 8, 9, and 14 (excluding §14.8 to §14.11). It will also cover additional material from the lectures, such as Sturm–Liouville theory, additional residue calculus techniques, and the Beltrami identity.

A note on convolutions

For the purposes of this course, the convolution of two functions f and g on $-\infty < x < \infty$ is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy. \quad (1)$$

For the specific case of dealing with Laplace transforms where functions are treated as $f(t) = 0$ for $t < 0$, this is equivalent to

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau. \quad (2)$$

The textbook presents Eq. 2 as the definition, but it should be viewed as a special case. Here, and in most other contexts, Eq. 1 should be viewed as the general definition.

Sample final questions

1. Consider

$$y''(x) + k^2y(x) = f(x)$$

where $k > 0$, on the interval $0 \leq x \leq \frac{\pi}{2k}$ subject to the boundary conditions $y(0) = y(\frac{\pi}{2k}) = 0$. Find a Green function solution of the form

$$y(x) = \int_{-\infty}^{\infty} G(x, x')f(x')dx'.$$

2. Let $z = x + iy$, and let a and b be positive real numbers. Calculate an equation in terms of x and y for the locus of points that satisfy

- (a) $\bar{r}z - r\bar{z} = 0$, where $r = a + ib$,
 (b) $(z^2 + \bar{z}^2)(b^2 - a^2) + 2z\bar{z}(b^2 + a^2) = 4a^2b^2$.

If $a = 2$ and $b = 1$, sketch the loci of points for the two cases.

3. Using Lagrange multipliers, find the maximum volume of a box described by $|x| < d$, $|y| < e$, and $|z| < f$ subject to the constraint

$$\frac{d^2}{a^2} + \frac{e^2}{b^2} + \frac{f^2}{c^2} \leq 1$$

where a , b , and c are arbitrary positive constants.

4. (a) Calculate the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

- (b) Show by explicit calculation that $A^2 = \beta A$ for some constant β , and hence show that $A^n = \beta^{n-1}A$ for any positive integer n .
 (c) The power series for the exponential is

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Calculate $\exp(\lambda A)$, expressing your answer in the form $f(\lambda)I + g(\lambda)A$ where f and g are real functions and I is the identity matrix.

5. For sufficiently small x , the Taylor series expansion of a function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}.$$

- (a) By using the definition above, calculate a complete Taylor series expansion for $\log(1 - x)$. Determine the precise interval of convergence of the series.
 (b) Calculate a power series expansion of $\log(1 - x^2 - x^3)$, keeping terms up to and including x^6 .
 6. By using residue calculus and a keyhole contour, or otherwise, evaluate the integral

$$\int_0^{\infty} \frac{x^{1/3}dx}{(x+2)^2}.$$

7. The parabolic coordinate system (a, b) can be defined as

$$x = ab, \quad y = \frac{a^2 - b^2}{2}.$$

For a given function $f(a, b)$, calculate expressions for $\partial_x f$ and $\partial_y f$ in terms of a , b , $\partial_a f$, and $\partial_b f$.

8. (a) Let the Laplace transform of a function $y(t)$ by $Y(p)$. Starting from the definition, calculate the Laplace transforms of y' and y'' in terms of $Y(p)$, $y(0)$, and $y'(0)$.

(b) Consider the differential equation

$$y'' + 9y' + 8y = 0$$

subject to the boundary conditions $y(0) = y'(0) = 1$. Calculate an expression for the Laplace transform of the solution, $Y(p)$.

(c) Use the Bromwich inversion integral

$$y(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} Y(p) dp$$

to calculate $y(t)$, where c is a positive constant.

9. (a) Show that if a function $f(x)$ has Fourier transform $\tilde{f}(\alpha)$, then the function $f'(x)$ has Fourier transform $i\alpha \tilde{f}(\alpha)$.

(b) Consider the differential equation

$$\frac{d^2 f}{dx^2} + \frac{df}{dx} - 2f = \delta(x),$$

subject to the boundary conditions $\lim_{x \rightarrow \pm\infty} |f(x)| = 0$. Calculate the Fourier transform of the solution, $\tilde{f}(\alpha)$.

(c) By applying residue calculus to the inverse Fourier transform, determine the solution $f(x)$.

10. Given a Laplace transform $F(p)$, the original function can be calculated using the Bromwich inversion integral

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p) e^{pt} dp$$

where c is a positive constant taken to be large enough that the contour passes to the right of any singularities. Using residue calculus, calculate the functions $f(t)$ for the cases of

$$F_1(p) = \frac{p+b}{(p+b)^2 + a^2}, \quad F_2(p) = \frac{2ap}{(p^2 + a^2)^2}$$

where a and b are real constants. Sketch the functions for the cases of $a = \pi$ and $b = 1/2$.

11. Use residue calculus to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)(x + i)} dx.$$

12. For a function f that is periodic on the interval $-\pi \leq x < \pi$, a smoothed function f_s can be constructed as

$$f_s(x) = \frac{1}{2l} \int_{x-l}^{x+l} f(y) dy$$

where $l > 0$. With this definition, the value of f_s at x is equal to average value of f over the range from $x - l$ to $x + l$.

- (a) Let c_n be the complex Fourier series coefficients of f , so that $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$. Calculate the complex Fourier series coefficients of f_s in terms of c_n .
- (b) Let a_n and b_n be the Fourier series coefficients of f , so that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

By direct calculation, or by using the results from part (a), determine the Fourier series coefficients of f_s in terms of a_n and b_n .

13. (a) Suppose that $z = x + iy$, and the complex function f is defined in terms of real and imaginary components as $u(x, y) + iv(x, y)$. By using the Cauchy–Riemann equations, $u_x = v_y$ and $u_y = -v_x$, determine whether the following functions are analytic:

- $f(x + iy) = x^3 + y^2x - i(yx^2 + y^3)$,
- $f(x + iy) = x^3 - 3y^2x + i(3yx^2 - y^3)$.

(b) Use the Cauchy–Riemann equations to show that

$$f(x, y) = e^{x^2 - y^2} (\cos 2xy + i \sin 2xy)$$

is analytic. Determine f as a function of z .

14. Consider building a tunnel from $(x, y) = (-a, 0)$ to $(x, y) = (a, 0)$ where $0 < a < 1$. Let the tunnel's shape be given by $y(x)$ where $y \leq 0$. If the cost per unit length of building the tunnel is proportional to $\sqrt{1 + y}$, determine the shape of the tunnel that gives a locally minimal cost. Show that for a fixed value of a , there are two tunnel shapes of locally minimal cost.

15. Consider the function

$$f(x, y) = x^2 - xy + 2y^2 - 3x - 2y.$$

Find the minimum and maximum values of f in

- (a) the triangle T_1 defined by $0 \leq y \leq x \leq 3$,
 (b) the triangle T_2 defined by $0 \leq x \leq y \leq 3$.

16. Use residue calculus to evaluate the integral

$$\oint_C \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

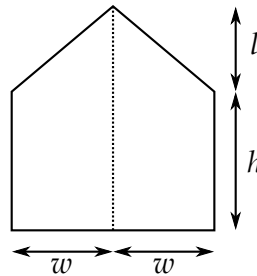
and hence show that

$$\int_{-\pi}^{\pi} \cos^{2n} t \, dt = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}.$$

Additional practice questions

The following questions are of a similar style and are good practice, although they involve too much algebra to be used on the exam.

17. Find the values of l , w , and h that maximize the area of the irregular pentagon shown below, subject to the constraint that the length of the perimeter is P .



18. Let

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

Sketch f and g . Calculate $(f * g)(x)$ for $x \geq 0$, and by symmetry deduce the value of $(f * g)(x)$ for $x < 0$. Show that $f * g$ is continuous and sketch it.

19. Repeat the previous exercise with an alternative form of g ,

$$g(x) = \begin{cases} x & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

20. Let a be a real number where $|a| < 1$. By starting from the geometric series formula, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, show that

$$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2}.$$

Hence determine the complex Fourier series of

$$f(x) = \frac{\lambda e^{ix}}{(1 - \lambda e^{ix})^2}$$

where $|\lambda| < 1$. Determine f_s as in defined in question 12.

21. The parabolic coordinate system (a, b) can be defined as

$$x = ab, \quad y = \frac{a^2 - b^2}{2}.$$

- Determine the radial distance $r = \sqrt{x^2 + y^2}$ in terms of a and b .
- Consider a particle of mass m whose position is described by $(a(t), b(t))$, moving in a potential $V = -\frac{m}{r}$. Calculate a Lagrangian $L(a, b, \dot{a}, \dot{b})$.
- Calculate the Euler–Lagrange equations for the a and b coordinates. Show that if C is a constant, then $a(t) = C$ is consistent with the equations, and solve for the corresponding $b(t)$ using the condition $b(0) = 0$. Express your answer as $t(b)$ where time is viewed as a function of b .