Math 121A: Midterm information

- The second midterm will take place in class on Wednesday, April 3rd from 3:10pm–4pm.
- The exam will cover Fourier series, Fourier transforms, Laplace transforms, solution of differential equations, seperable solutions to PDEs, and Sturm–Liouville theory. This corresponds to chapters 7 and 8 in the textbook, plus some of the additional material covered in the lectures.
- The exam is closed book – no textbooks, notebooks, or calculators allowed.
- You will be expected to know the basic definitions, such as the definition the Fourier series, complex Fourier series, Parseval's theorem, convolution, Fourier transform, and Laplace transform. However, it will not be necessary to memorize any tables of results of Laplace and Fourier transforms—any that are needed will be given in the question.
- There will be four questions, each of which will be graded out of ten points.

Sample midterm questions

1. Consider
$$y''(x) - k^2 y(x) = f(x)$$
for $-\infty < x < \infty$, subject to the boundary conditions
$$\lim_{x\to-\infty} y(x) = \lim_{x\to\infty} y(x) = 0.$$
Find a Green function solution of the form
$$y(x) = \int_{-\infty}^{\infty} G(x,x')f(x')dx'.$$

2. Define $h_\lambda(x) = e^{-\lambda x^2}$ for any $\lambda > 0$.
   (a) Calculate the Fourier transform $\tilde{h}_\lambda$ of h_λ.
   (b) Calculate the convolution $g = h_\lambda * h_\mu$.
   (c) For the case of $\lambda = 1$ and $\mu = 2$, sketch h_λ, h_μ, and g.
   (d) Verify that the Fourier transform of g is equal to $2\pi\tilde{h}_\lambda\tilde{h}_\mu$.

   You may use the result that
$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}.$$

3. (a) By using Fourier and/or Laplace transforms, solve the partial differential equation

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = b\frac{\partial^2 f}{\partial x^2}$$

for the function $f(x,t)$, where $b > 0$ and the initial condition is $f(x,0) = \delta(x)$. You may use the result that

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}.$$

(b) Write down a Green function solution to the problem

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = b\frac{\partial^2 f}{\partial x^2}$$

subject to the initial condition $f(x,0) = g(x)$.

(c) Explicitly evaluate the solution in part (b) for the case when $g(x) = e^{-ax}$, and check that it satisfies the partial differential equation and initial condition.

4. (a) Calculate the Fourier series of

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0, \\ 1 & \text{for } 0 \leq x < \pi. \end{cases}$$

(b) By using Parseval's theorem, show that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

5. Use Laplace transforms to solve the differential equation

$$y'' + 6y' + 8y = e^{-3t}$$

subject to the boundary conditions $y(0) = 1$ and $y'(0) = 0$.