# Math 121A: Midterm 2 solutions

1. Consider the differential equation

$$y'' + 11y' + 30y = 0$$

for the function $y(t)$ on the range $0 \le t < \infty$.

(a) Calculate the Laplace transform of the function $f(t) = e^{-at}$.

**Answer:** The Laplace transform is

$$F(p) = \int_0^\infty f(t)e^{-pt}dt = \int_0^\infty e^{-(p+a)t}dt = \left[\frac{-e^{-(p+a)t}}{p+a}\right]_0^\infty = \frac{1}{p+a}.$$

(b) Determine $y(t)$ using Laplace transforms for the conditions $y(0) = 0$, $y'(0) = 1$.

**Answer:** Taking the Laplace transform of the equation gives

$$\left[p^2Y(p) - py(0) - y'(0)\right] + 11\left[pY(p) - y(0)\right] + 30Y(p) = 0$$

and hence

$$Y(p) = \frac{1}{p^2 + 11p + 30} = \frac{1}{(p+5)(p+6)} = \frac{1}{p+5} - \frac{1}{p+6}.$$

Using the result from part (a),

$$y(t) = e^{-5t} - e^{-6t}.$$

(c) Determine $y(t)$ using Laplace transforms for the conditions $y(0) = 1$, $y'(0) = 0$.

**Answer:** By reference to part (b), with the given boundary conditions,

$$Y(p) = \frac{p+11}{(p+5)(p+6)} = \frac{1}{p+5} + \frac{5}{(p+5)(p+6)} = \frac{1}{p+5} + \frac{5}{p+5} - \frac{5}{p+6}.$$

Hence by using part (a),

$$y(t) = 6e^{-5t} - 5e^{-6t}.$$

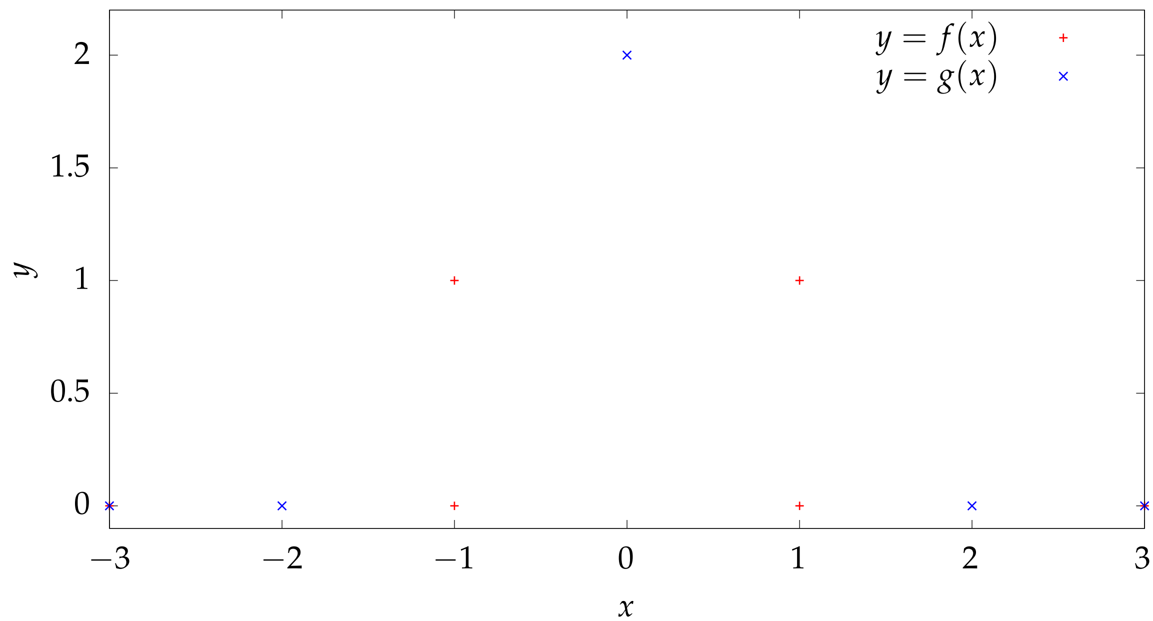


Figure 1: Plots of the functions $f$ and $g$ considered in question 2.

2. Consider the function

$$f(x) = \begin{cases} 1 & \text{for } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the convolution $g = f * f$. Sketch $f$ and $g$.

**Answer:** The convolution is given by

$$g(x) = \int_{-\infty}^{\infty} f(\xi) f(x - \xi) d\xi.$$

For $0 \leq x < 2$ this is

$$g(x) = \int_{x}^{2} 1 \, d\xi = 2 - x$$

and for $x \geq 2$, $g(x) = 0$. Since $g$ is even it follows that

$$g(x) = \begin{cases} 2 - |x| & \text{for } |x| < 2, \\ 0 & \text{for } |x| \geq 2. \end{cases}$$

Plots of $f$ and $g$ are shown in Fig. 1.

(b) Determine the Fourier transform of $f$.

**Answer:** The Fourier transform is

$$\begin{aligned}\tilde{f}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ix\alpha}dx = \frac{1}{2\pi}\int_{-1}^{1} e^{-ix\alpha}dx \\ &= \left[\frac{e^{-ix\alpha}}{-2\pi i\alpha}\right]_{-1}^{1} = \frac{e^{-i\alpha} - e^{i\alpha}}{-2\pi i\alpha} = \frac{\sin\alpha}{\pi\alpha}.\end{aligned}$$

(c) Determine the Fourier transform of $g$ either by direct calculation, or by making use of standard results and your answer from part (b).

**Answer:** By making use of the standard result for convolutions, the Fourier transform is given by

$$\tilde{g}(\alpha) = 2\pi\tilde{f}(\alpha)\tilde{f}(\alpha) = \frac{2\sin^2\alpha}{\pi\alpha^2}.$$

3. Consider the differential equation

$$y'' = f(x)$$

subject to $y(0) = 0$, and $y'(1) = 0$.

(a) Calculate a Green function solution of the form

$$y(x) = \int_0^1 G(x, x') f(x') dx'.$$

**Answer:** Consider solving the equation for an impulsive input, $f(x) = \delta(x - x')$. In the regions $x > x'$ and $x < x'$ the solution has the form

$$y(x) = Ax + B.$$

To satisfy the boundary conditions, it must be $y(x) = Ax$ for $x < x'$ and $y(x) = B$ for $x > x'$. To ensure continuity, $Ax' = B$. To ensure $y'_+(x') - y'_-(x') = 1$, $0 - A = 1$. Hence $A = -1$ and $B = -x'$, and the Green function is

$$G(x, x') = \begin{cases} -x & \text{for } x < x', \\ -x' & \text{for } x > x'. \end{cases}$$

(b) Explicitly calculate the solution $y(x)$ for the case when $f(x) = x$ and check that this solution satisfies the differential equation and the boundary conditions.

**Answer:** The solution is given by

$$\begin{aligned} y(x) &= \int_0^1 G(x, x') x' \, dx' \\ &= -\int_0^x x'^2 \, dx' - \int_x^1 x x' \, dx' \\ &= -\left[\frac{x'^3}{3}\right]_0^x - x\left[\frac{x'^2}{2}\right]_x^1 \\ &= -\frac{x^3}{3} + \frac{x^3}{2} - \frac{x^2}{2} = \frac{x^3}{6} - \frac{x}{2}. \end{aligned}$$

It can be seen that

$$y'(x) = \frac{x^2}{2} - \frac{1}{2}, \qquad y''(x) = x$$

and thus $y(0) = 0$, $y'(1) = 0$, and $y''(x) = f(x)$.

4. (a) Calculate the Fourier series of

$$f(x) = \begin{cases} a - |x| & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

over the range $-\pi < x < \pi$, where $0 \leq a < \pi$.

**Answer:** Since $f$ is even

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^a (a-x)\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^a \frac{\sin nx}{n}dx + \left[(a-x)\frac{\sin nx}{n}\right]_0^a \\ &= \frac{2}{\pi}\left[-\frac{\cos nx}{n^2}\right]_0^a = \frac{2(1-\cos na)}{\pi n^2} \end{aligned}$$

and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^a (a-x)dx = \frac{a^2}{\pi}.$$

Hence

$$f(x) = \frac{a^2}{2\pi} + \sum_{n=1}^{\infty} \frac{2(1-\cos na)\cos nx}{\pi n^2}.$$

(b) By considering Parseval's theorem and a suitable choice of $a$, show that

$$\sum_{n=1}^{\infty} \frac{\sin^4 n}{n^4} = \frac{\pi}{3} - \frac{1}{2}.$$

**Answer:** Integrating the square of the function gives

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{2}{\pi}\int_0^a (x-a)^2 dx = \frac{2a^3}{3\pi}$$

and by Parseval's theorem this is equal to

$$\frac{a_0^2}{2} + \sum_{n=1}^{\pi} a_n^2 = \frac{a^4}{2\pi^2} + \sum_{n=1}^{\infty} \frac{4(1-\cos na)^2}{\pi^2 n^4}.$$

To obtain the given equality, consider $a = 2$, to give

$$\frac{16}{3\pi} = \frac{8}{\pi^2} + \sum_{n=1}^{\infty} \frac{4(1-\cos 2n)^2}{\pi^2 n^4}.$$

Using a half-angle formula $(1-\cos 2n) = 2\sin^2 n$ this can be written as

$$\frac{\pi}{3} - \frac{1}{2} = \sum_{n=1} \frac{\sin^4 n}{n^4}.$$