Math 121A: Midterm 1 solutions

1. The function
$$f(x,y,z) = x^2 + 3y^2 + 5z^2 + 2xy + 4yz - 4x - 2z$$
has one minimum point. Find its location.

Answer: At the minimum point, all the partial derivatives of f must vanish, and hence

$$\begin{aligned} 0 &= f_x &= 2x + 2y - 4 \\ 0 &= f_y &= 2x + 6y + 4z \\ 0 &= f_z &= 4y + 10z - 2. \end{aligned}$$

These equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

To solve this system, an augmented matrix can be constructed, which can then be row reduced according to

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 0 & 2 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and hence $x = 4$, $y = -2$, and $z = 1$.

2. (a) Calculate the derivative of $f(x) = \log(\log(\log(x)))$.

Answer: The derivative is

$$f'(x) = \frac{1}{\log(\log(x))\log(x)x}.$$

(b) By using an appropriate series test, determine whether

$$\sum_{n=3}^{\infty} \frac{1}{n\log(n)\log(\log(n))}$$

converges or diverges.

Answer: By reference to the previous answer, it can be seen that

$$\int \frac{1}{n\log(n)\log(\log(n))} = \log(\log(\log(n))) + C$$

and since $\log n \to \infty$ as $n \to \infty$, it follows that $\log(\log(\log(n))) \to \infty$ as $n \to \infty$, and hence the series must diverge by the integral test.

(c) By using an appropriate series test, determine whether

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n\log(n)\log(\log(n))}$$

converges or diverges.

Answer: The coefficients can be written as $(-1)^n a_n$ where $a_n = f'(n)$ as defined above. Since $\lim_{n\to\infty} \log n = \infty$ it follows that $\lim_{n\to\infty} a_n = 0$. It can also be seen that $a_n > 0$ for all $n \geq 3$. Hence the series satisfies the conditions of the alternating series test, and therefore it must converge.

3. By using Lagrange multipliers, find the smallest possible surface area (including both ends) of a cylinder with volume V.

Answer: Let the radius and height of the cylinder be r and h respectively. Its volume is $V = \pi r^2 h$, and its surface area is $S = 2\pi r^2 + 2\pi rh$. To maximize the surface area for a fixed volume, consider the augmented function

$$S(r, h, \lambda) = 2\pi r^2 + 2\pi rh + \lambda(\pi r^2 h - V)$$

where λ is a Lagrange multiplier. Taking partial derivatives and setting them to zero gives

$$\begin{aligned} 0 &= S_r = 4\pi r + 2\pi h + 2\lambda\pi rh \\ 0 &= S_h = 2\pi r + \lambda\pi r^2. \end{aligned}$$

Rearrangement of the second equation gives

$$\lambda = -\frac{2}{r}$$

whereupon substitution into the first equation gives

$$4\pi r + 2\pi h = 4\pi h$$

and hence $2r = h$. Therefore $V = 2\pi r^3$ and hence

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

The surface area is then

$$S = 2\pi r^2 + 4\pi r^2 = 6\pi r^2 = 3\sqrt[3]{2\pi V^2}.$$

4. (a) By considering appropriate powers of $e^{i\theta} = \cos\theta + i\sin\theta$ or otherwise, determine an expression for $\sin^3\theta$ as a linear combination of terms with the form $\sin k\theta$.

Answer: Note that

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= e^{3i\theta} = (e^{i\theta})^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.\end{aligned}$$

Equating imaginary parts gives

$$\begin{aligned}\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta \\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

and hence $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

(b) Consider the annulus A defined as $a \leq r \leq b$ in polar coordinates, where $0 < a < b$. Show that for any integer k, the function $r^{\pm k}\sin k\theta$ is a solution to the Laplace equation $\nabla^2\phi = 0$ in A.

Answer: The functions satisfy

$$\begin{aligned}\nabla^2(r^{\pm k}\sin k\theta) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}r^{\pm k}\right)\sin k\theta + r^{\pm k-2}\frac{\partial^2}{\partial\theta^2}(\sin k\theta) \\ &= \pm\frac{k}{r}\frac{\partial}{\partial r}\left(r^{\pm k}\right)\sin k\theta - k^2 r^{\pm k-2}\sin k\theta \\ &= k^2 r^{\pm k-2}\sin k\theta - k^2 r^{\pm k-2}\sin k\theta = 0.\end{aligned}$$

(c) Find a solution to $\nabla^2\phi = 0$ in A that satisfies the boundary conditions

$$\phi(a,\theta) = 4\sin^3\theta, \qquad \phi(b,\theta) = 0.$$

Answer: Since the condition at $r = a$ can be written as $\phi(a,\theta) = 3\sin\theta - \sin 3\theta$, a solution can be searched for with the form

$$\phi(r,\theta) = (Ar + Br^{-1})\sin\theta + (Cr^3 + Dr^{-3})\sin 3\theta.$$

To agree with the boundary condition at $r = a$,

$$Aa + \frac{B}{a} = 3, \qquad Ca^3 + \frac{D}{a^3} = -1$$

and to agree with the boundary condition at $r = b$,

$$Ab + \frac{B}{b} = 0, \qquad Cb^3 + \frac{D}{b^3} = 0.$$

Substitution gives

$$Aa - \frac{Ab^2}{a} = 3, \qquad Ca^3 - \frac{Cb^6}{a^3} = -1,$$

and hence

$$A = \frac{3a}{a^2 - b^2}, \qquad C = \frac{a^3}{b^6 - a^6}.$$

Therefore

$$B = \frac{3b^2 a}{b^2 - a^2}, \qquad D = \frac{a^3 b^6}{a^6 - b^6}.$$

and thus the solution is

$$\phi(r,\theta) = \frac{3a(r^2 - b^2)}{r(a^2 - b^2)} \sin\theta - \frac{a^3(r^6 - b^6)}{r^3(a^6 - b^6)} \sin 3\theta.$$