Math 121A: Midterm 2

1. Consider the differential equation

$$y'' + 11y' + 30y = 0$$

for the function $y(t)$ on the range $0 \leq t < \infty$.

(a) Calculate the Laplace transform of the function $f(t) = e^{-at}$.

(b) Determine $y(t)$ using Laplace transforms for the conditions $y(0) = 0$, $y'(0) = 1$.

(c) Determine $y(t)$ using Laplace transforms for the conditions $y(0) = 1$, $y'(0) = 0$.

2. Consider the function

$$f(x) = \begin{cases} 1 & \text{for } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the convolution $g = f * f$. Sketch f and g.

(b) Determine the Fourier transform of f.

(c) Determine the Fourier transform of g either by direct calculation, or by making use of standard results and your answer from part (b).

3. Consider the differential equation

$$y'' = f(x)$$

on the range $0 \leq x \leq 1$ subject to $y(0) = 0$ and $y'(1) = 0$.

(a) Calculate a Green function solution of the form

$$y(x) = \int_0^1 G(x, x') f(x') dx'.$$

(b) Explicitly calculate the solution $y(x)$ for the case when $f(x) = x$ and check that this solution satisfies the differential equation and the boundary conditions.

4. (a) Calculate the Fourier series of

$$f(x) = \begin{cases} a - |x| & \text{for } |x| < a, \\ 0 & \text{for } |x| \geq a, \end{cases}$$

over the range $-\pi < x < \pi$, where $0 \leq a < \pi$.

(b) By considering Parseval's theorem and a suitable choice of a, show that

$$\sum_{n=1}^{\infty} \frac{\sin^4 n}{n^4} = \frac{\pi}{3} - \frac{1}{2}.$$