Math 121A: Homework 7 (due March 20)

1. Consider solving the wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$$

for $f(x,t)$ in the interval $-\pi \le x \le \pi$, subject to the boundary conditions

$$\left.\frac{\partial f}{\partial x}\right|_{x=-\pi} = \left.\frac{\partial f}{\partial x}\right|_{x=\pi} = 0$$

and initial conditions

$$f(x,0) = g(x) = (|x| - \pi)^2, \qquad \left.\frac{\partial f}{\partial t}\right|_{t=0} = 0.$$

   (a) Find the Fourier series of $g(x)$.

   (b) Search for separable solutions of the form $f(x,t) = X(x)T(t)$ where $X(x)$ is even.

   (c) By using parts (a) and (b), write down a general solution for $f(x,t)$ in terms of an infinite series.

   (d) The kinetic energy and potential energy for the system can be defined as

$$K(t) = \frac{1}{2}\int_{-\pi}^{\pi} \left(\frac{\partial f}{\partial t}\right)^2 dx, \qquad P(t) = \frac{1}{2}\int_{-\pi}^{\pi} c^2 \left(\frac{\partial f}{\partial x}\right)^2 dx.$$

   respectively. Let the total energy be given by $E(t) = K(t) + P(t)$. By considering the time derivative of E and making use of integration by parts, show that the total energy is constant.

   (e) By using the initial conditions for f and $\partial f / \partial t$ calculate $E(0)$.

   (f) Calculate $E(t)$ using the series solution from part (c), and show that it is constant.

   (g) **Optional for the enthusiasts.** Use a computer to plot $f(x,t)$ over the range from $t = 0$ to $t = 2\pi/c$.

2. (a) Calculate the Fourier transform $\tilde{f}(\alpha)$ of the function

$$f(x) = \begin{cases} \cos x & \text{for } |x| < \pi/2, \\ 0 & \text{for } |x| \ge \pi/2. \end{cases}$$

   (b) Calculate the Fourier transform $\tilde{g}(\alpha)$ of the function

$$g(x) = \begin{cases} \sin x & \text{for } 0 < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

(c) What is $\tilde{f}(\alpha)/\tilde{g}(\alpha)$ and why should this be expected?

(d) By using the previous answers and using basic properties of Fourier transforms, without doing any integration, determine the Fourier transform of

$$h(x) = \begin{cases} \sin x & \text{for } -\pi < x < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

(e) **Optional for the enthusiasts.** Without doing any integration, calculate the Fourier transform of

$$q_n(x) = \begin{cases} \sin x & \text{for } -n\pi < x < n\pi, \\ 0 & \text{otherwise.} \end{cases}$$

where n is a positive integer. Plot the imaginary component of $\tilde{q}_n(\alpha)$ over the range $-2 \leq \alpha \leq 2$ for the cases of $n = 10, 20, 30$ and interpret the shapes of the graphs.

3. In the United Kingdom people greatly enjoy drinking tea, particularly with a splash of milk. However, there is often some debate about the correct procedure for adding the milk.

   Suppose that the tea is initially at $T_t = 95\ ^\circ\text{C}$, the room temperature is $T_r = 20\ ^\circ\text{C}$, and the milk is kept in a refrigerator at $T_m = 5\ ^\circ\text{C}$. Let the volume of the tea be $V_t = 200$ ml and the volume of the milk be $V_m = 50$ ml. Assume that the rate of change in the tea's temperature is given by λ multiplied by the difference between the tea's temperature and T_r. Derive a differential equation for the temperature of the tea over time.

   Suppose now that $\lambda = (\log 2)/5 \text{ min}^{-1} = 0.1386 \text{ min}^{-1}$. Determine the temperature of the tea after the following two alternative procedures:

   - The milk is added to the tea, and the tea is left to stand for 20 minutes.
   - The tea is left to stand for 20 minutes, and then the milk is added.

   After which procedure is the tea hotter?

4. Boas exercise 8.4.16

5. Boas exercise 8.4.20