Math 121A: Homework 6 (due March 13)

1. Consider the function $f(x) = x(\pi - x)$ defined on $0 \leq x \leq \pi$.
   (a) Calculate the Fourier sine series f_s and Fourier cosine series f_c of f.
   (b) For both f_s and f_c, plot the sum of the first four non-zero terms over the range $-\pi \leq x < \pi$ and compare with the exact solution. Which series is more accurate and why?
   (c) By evaluating $f_s(\frac{\pi}{2})$ and $f_c(\frac{\pi}{2})$, find two expressions for π^3 and π^2 as infinite series of fractions.
   (d) By using Parseval's theorem find expressions for π^6 and π^4 as infinite series of fractions.

2. Boas exercise 7.7.12

3. Boas exercise 7.7.13

4. Let $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$. Suppose that $f'(x) = \sum_{-\infty}^{\infty} d_n e^{inx}$ and $f(x-l) = \sum_{-\infty}^{\infty} q_n e^{inx}$. Express the coefficients d_n and q_n in terms of c_n.

5. Let f and g be periodic functions on the interval $-\pi \leq x < \pi$, with complex Fourier series
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad g(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx}.$$
   Let $f * g$ be the *convolution* of f and g, defined as
$$(f * g)(x) = \int_{-\pi}^{\pi} f(y) g(x-y) dy.$$
   Calculate the complex Fourier series coefficients of $f * g$ in terms of c_n and d_n.

6. (a) Calculate the Fourier series of the function defined on $-\pi \leq x < \pi$ as
$$f(x) = \begin{cases} 1 & \text{for } |x| < \pi/2, \\ 0 & \text{for } |x| \geq \pi/2. \end{cases}$$
   (b) Let $f_\lambda(x)$ be a filtered version of f, in which the higher frequency components are damped: if f has Fourier coefficients a_n and b_n, then let f_λ have coefficients $\lambda^n a_n$, $\lambda^n b_n$. Plot f, and f_λ for the cases of $\lambda = 0.7, 0.8, 0.9$.
   (c) By making use of the result from question 5, find a function $K_\lambda(x)$ such that $f_\lambda = K_\lambda * f$ for any function f. Plot K_λ for $\lambda = 0.7, 0.8, 0.9$.