Math 121A: Homework 5 (due March 6)

Office hours: Monday March 4, 4PM–6PM.

Part 1: Multiple integration

1. Boas exercise 5.3.9
2. Boas exercise 5.4.7
3. Boas exercise 5.4.16
4. (a) Calculate the volume of the three-dimensional object given by the points (x, y, z) that satisfy $x^2 + y^2 \leq 1$ and $x^2 + z^2 \leq 1$.
   (b) **Optional for the enthusiasts.** Calculate the volume of the three-dimensional object given by $x^2 + y^2 \leq 1$, $x^2 + z^2 \leq 1$, $y^2 + z^2 \leq 1$.

Part 2: Sturm–Liouville theory

Many physical problems, such as the drum vibration example considered in class, result in solving differential equations with the form

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = \lambda w(x)y$$

on an interval $a \leq x \leq b$, where $p(x) > 0$ and $w(x) > 0$ for all x. This is coupled with boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0$$

with the restrictions that $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$; this restriction ensures that both of the α_i and β_i cannot be zero, which would result in a meaningless boundary condition. The above problem can also be written as

$$Ly = \lambda y$$

where λ is viewed as an eigenvalue and L is a linear differential operator defined by

$$L\phi = \frac{1}{w(x)}\left(-\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi\right)$$

for any function ϕ. A problem of this form is referred to as a *Sturm–Liouville problem*, and its solutions have a number of special properties that can be mathematically proved:

- There is an infinite set of real eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$ whose values are unbounded.

- For each eigenvalue, there is a unique eigenfunction solution y_n that satisfies $Ly_n = \lambda_n y_n$, that has exactly $n-1$ zeroes on the interval $a < x < b$.
- If an inner product for two functions f and g is defined as

  $$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx,$$

  then, for suitable normalization, the eigenfunctions form an orthonormal basis, so that $\langle y_m, y_n \rangle = \delta_{mn}$.

These properties can extremely useful in constructing an evaluating solutions. In the vibrating drum example considered in class, searching for a separable solution leads to a Bessel equation in the radial direction, with boundary conditions that make it into a Sturm–Liouville problem.

Exercises

5. Find the eigenvalues and corresponding eigenfunctions for the Sturm–Liouville problem

   $$\frac{d}{dx}\left[-e^{-2x}\frac{dy}{dx}\right] - e^{-2x}y = \lambda e^{-2x} y$$

   on the interval $0 \leq x \leq \pi$, subject to the boundary conditions

   $$y(0) = 0, \qquad y'(\pi) - y(\pi) = 0.$$

   Verify that the eigenfunctions satisfy the orthogonality relation, so that $\langle y_m, y_n \rangle = 0$ for any $m \neq n$.

6. The concentration of gas in a channel over the range $0 < x < a$ follows the diffusion equation

   $$\frac{\partial f}{\partial t} = b \frac{\partial^2 f}{\partial x^2}.$$

   At $x = 0$ any gas is removed, and at $x = a$ there are is a wall that is impermeable to gas, so suitable boundary conditions are

   $$f(0, t) = 0, \qquad \frac{\partial f}{\partial x}(a, t) = 0.$$

   (a) Solve the system by searching for separable solutions of the form $f(x, t) = X(x)T(t)$. (This will result in a Sturm–Liouville problem for X.)

   (b) Write down the solution $f(x, t)$ to the equation for the initial condition

   $$f(x, 0) = \sin \frac{\pi x}{2a} + \frac{1}{5} \sin \frac{17\pi x}{2a}$$

   For the case of $a = \pi$ and $b = 1$, plot $f(x, t)$ for the values of t of 0, 0.2, 0.5, 1.0, 2.0, and 5.0. Interpret the graphs physically.