Math 121A: Homework 2 (due February 6)

Office hours: Monday Feb 4, 4PM–5PM, by Dr. Matthias Morzfeld, Evans 971.

1. Calculate the Taylor series at $x = 0$ of the following functions up to the term in x^3.
   (a) $e^x \sin x$
   (b) $\frac{1}{1+x+x^2}$
   (c) $\sin(\log(1+x))$

2. By considering Taylor's theorem with remainder, show that the Taylor series for $\sin x$ converges for all x.

3. Boas exercise 1.15.33

4. (a) The Taylor series for arctangent is

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots.$$

   By evaluating the first eight terms in the series for $x = 1$, calculate an approximation for π.

   (b) **Optional for the enthusiasts.** Use a computer to approximate π using the first million terms of the series in (a).

   (c) Calculate $(3+i)^2(7+i)$ and use your result to deduce that

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}.$$

   (d) By considering the first four terms the Taylor series for each arctangent, determine an approximation for π, and compare its accuracy with part (a).

5. The complex numbers z and w are related by

$$w = \frac{1+iz}{i+z}.$$

   Write $z = x + iy$ and $w = u + iv$ where u, v, x, and y are real.

   (a) Find expressions for u and v in terms of x and y.

   (b) By writing $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0, -\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$, with one point omitted.