Math 121A: Homework 11 (due May 3)

Note: since this is the last week of class, this homework is due on Friday.

1. Boas exercise 9.2.5

2. (a) In polar coordinates, the total length of a curve $r(\theta)$ is given by the functional

$$I[r] = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

By using the Euler equation, derive a differential equation for geodesics in polar coordinates. *(Note: the question asks you derive the differential equation, but not to solve it.)*

(b) Consider the a straight line from the points $(x, y) = (1, -1)$ to $(x, y) = (1, 1)$. Write the line as a function $r(\theta)$ and show that it satisfies the differential equation from part (a).

3. (a) By using the Euler equation, derive a differential equation for the function $y(x)$ the minimizes the distance

$$\int_1^2 g(x) ds = \int_1^2 g(x)\sqrt{1 + y'^2}\, dx$$

where $g(x)$ is an arbitrary function.

(b) Using the boundary conditions $y(1) = 0$ and $y(2) = 2$, solve the differential equation for the two cases of $g(x) = 1$ and $g(x) = \sqrt{x}$.
[*Hint: for the second case, the solution has the form* $y(x) = a(x - b)^c$.]

(c) Plot the two solutions $y(x)$, and discuss how their shapes correspond to what is being minimized.

4. (a) For a function $y(x)$ defined on an interval $x_1 \le x \le x_2$, consider the case of minimizing the functional

$$I[y] = \int_{x_1}^{x_2} F(x, y, y', y'') dx.$$

By following the original Euler equation derivation, and assuming zero values of the variation $\eta(x)$ and its derivative at x_1 and x_2, show that in this case the Euler equation becomes

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0.$$

(b) Find the solution $y(x)$ on the interval $0 \leq x \leq 1$ that minimizes

$$I[y] = \frac{1}{2} \int_0^1 (y'')^2 dx$$

subject to the boundary conditions $y(0) = 1$, $y(1) = -1$, and $y'(0) = y'(1) = 0$. Calculate the value of $I[y]$.

(c) Consider the function $y_*(x) = \cos \pi x$. Show that it satisfies the boundary conditions of part (b). Calculate the value of $I[y_*]$ and compare its numerical value with $I[y]$ from part (b).

5. Boas exercise 9.6.1

6. Consider a particle of mass m sliding on a frictionless vertical hoop of radius a, whose position is described by the angle $\theta(t)$ that it makes with the negative y axis, so that $(x, y) = (a \sin \theta, -a \cos \theta)$. Assume that gravity points in the negative y direction.

   (a) Write down a Lagrangian for the system, in terms of the kinetic energy minus the potential energy, and hence derive a differential equation for $\theta(t)$ describing the particle's position.

   (b) Solve the equation for the case when θ is small, corresponding to the case of small oscillations about the vertical axis. Find the frequency of oscillations.

   (c) **Optional for the enthusiasts.** What physical principle does the Beltrami identity correspond to?