Math 121A: solutions to final exam

1. (a) Calculate the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Answer: The eigenvalues are solutions to

$$\begin{aligned} 0 &= \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda)^2 - 2(1-\lambda) = (1-\lambda)(-2-\lambda+\lambda^2) = (1-\lambda)(2-\lambda)(1+\lambda) \end{aligned}$$

and hence $\lambda = 1, -1, 2$. Write $\mathbf{u}_\lambda = (u_\lambda, v_\lambda, w_\lambda)$ for the eigenvector corresponding to λ. For $\lambda = 1$ the eigenvector satisfies

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and thus $\mathbf{u}_1 = (-1, 1, 0)$ is a solution. The other two eigenvectors are given by

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_{-1} \\ v_{-1} \\ w_{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and thus $\mathbf{u}_{-1} = (1, 1, -2)$ and $\mathbf{u}_2 = (1, 1, 1)$ are solutions.

(b) Solve the linear system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}.$$

Answer: Note that the right hand side is $2\mathbf{u}_2 + 2\mathbf{u}_1$. This is equal to $A(\mathbf{u}_2 + 2\mathbf{u}_1)$ and thus $(x, y, z) = (-1, 3, 1)$.

2. (a) Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(-3)^n}.$$

By considering the series for $x = \pm R$ and using appropriate series tests, determine the exact range of x for which it converges.

Answer: If the series is written as $\sum a_n$, then the radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-3)^n n}{(-3)^{n+1}(n+1)} \right| = \lim_{n\to\infty} \left| \frac{n}{-3(n+1)} \right| = \frac{1}{3}$$

and hence $R = 3$. For $x = 3$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the alternating series test. For $x = -3$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges by the integral test, since $\int n^{-1} dn = C + \log n$ which tends to infinity as $n \to \infty$. Hence the series converges for $-3 < x \leq 3$.

(b) For the function $f(t) = -5 + 2t^2$, calculate $f(0)$, $f(1)$, $f(2)$, and $f(3)$. Use the results to sketch f over the range $-3 \leq t \leq 3$.

Answer: The function values are $f(0) = -5$, $f(1) = -3$, $f(2) = 3$, $f(3) = 13$. A graph is shown in Fig. 1.

(c) Determine the precise set of values of t for which the series

$$\sum_{n=1}^{\infty} \frac{[f(t)]^n}{n(-3)^n}$$

will converge.

Answer: By reference to part (a), it can be seen that the series will converge for $-3 < f(t) \leq 3$. By reference to the graph from part (b), this will correspond to $-2 \leq t < -1$ and $1 < t \leq 2$.

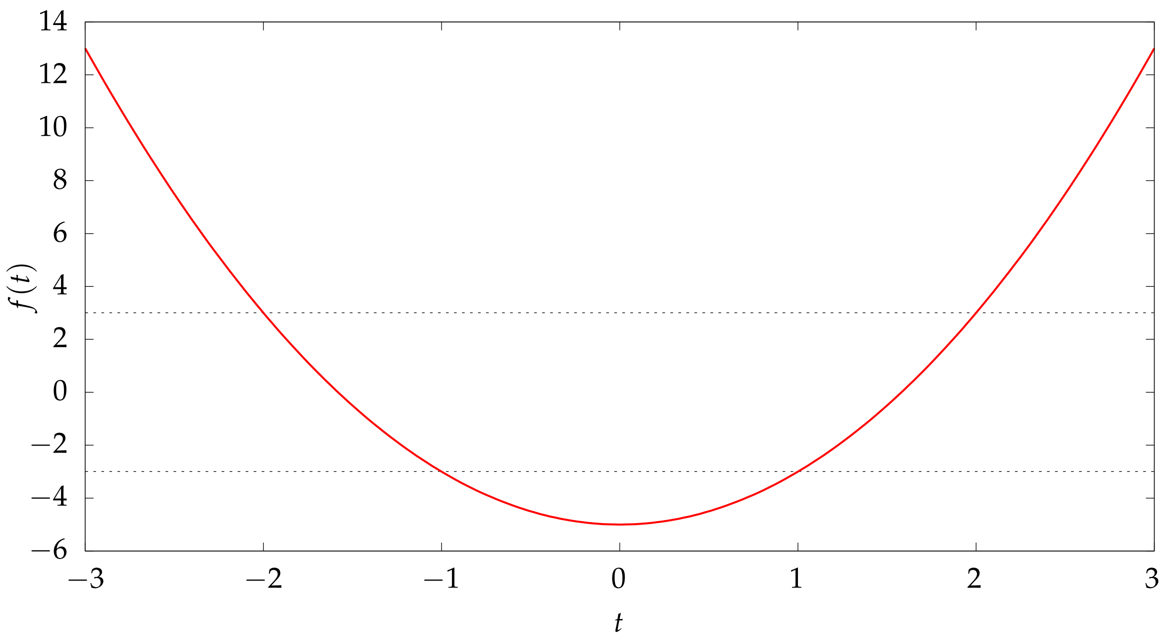

Figure 1: Quadratic function considered in question 2. The dashed lines correspond to $f(t) = \pm 3$, marking the limits of the values where the series in part (c) will converge.

3. (a) Let a function $f(x)$ have the Fourier transform $\tilde{f}(\alpha)$. Let $g(x) = f(-x)$ and $h(x) = xf(x)$. Show that the Fourier transforms of g and h are given by $\tilde{g}(\alpha) = \tilde{f}(-\alpha)$ and $\tilde{h}(\alpha) = i\tilde{f}'(\alpha)$.

Answer: The Fourier transform of g is

$$\begin{aligned}\tilde{g}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-x)e^{-i\alpha x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{i\alpha(-y)}dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{-i(-\alpha)y}dy = \tilde{f}(-\alpha).\end{aligned}$$

and the Fourier transform of h is

$$\begin{aligned}\tilde{h}(\alpha) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} xf(x)e^{-i\alpha x}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)(xe^{-i\alpha x})dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)i\frac{\partial}{\partial\alpha}(e^{-i\alpha x})dx = \frac{i}{2\pi}\frac{d}{d\alpha}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx = i\frac{d\tilde{f}}{d\alpha}.\end{aligned}$$

(b) Determine the Fourier transform $\tilde{f}(\alpha)$ of the function

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Answer: The Fourier transform is given by

$$\tilde{f}(\alpha) = \frac{1}{2\pi}\int_0^{\infty} e^{-x}e^{-i\alpha x}dx = \frac{1}{2\pi}\left[\frac{e^{-(1+i\alpha)x}}{-(1+i\alpha)}\right]_0^{\infty} = \frac{1}{2\pi(1+i\alpha)}.$$

(c) By using the above results, or otherwise, determine the Fourier transform $\tilde{s}(\alpha)$ of

$$s(x) = x^2 e^{-|x|}.$$

Show that $\tilde{s}(\alpha)$ is real.

Answer: It can be seen that $s(x) = x^2(f(x) + f(-x))$. By using the results of part (a), it follows that

$$\begin{aligned}
\tilde{s}(\alpha) &= \left(i\frac{d}{d\alpha}\right)(f(\alpha) + f(-\alpha)) = -\frac{1}{2\pi}\frac{d^2}{d\alpha^2}\left(\frac{1}{1+i\alpha} + \frac{1}{1-i\alpha}\right) \\
&= -\frac{1}{2\pi}\left(\frac{2i^2}{(1+i\alpha)^3} + \frac{2i^2}{(1-i\alpha)^3}\right) = \frac{(1-i\alpha)^3 + (1+i\alpha)^3}{\pi(1-i\alpha)^3(1+i\alpha)^3} \\
&= \frac{2-6\alpha^2}{\pi(1+\alpha^2)^3}
\end{aligned}$$

and since $\tilde{s}(\alpha)$ has no terms involving i it is real.

4. (a) Calculate the complex Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for the function

$$f(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0 \end{cases}$$

defined on the interval $-\pi \leq x < \pi$.

Answer: For $n = 0$, the coefficient is

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{2\pi}\int_{-\pi}^{0}(-1)dx + \frac{1}{2\pi}\int_{0}^{\pi}(1)dx = -\frac{1}{2} + \frac{1}{2} = 0.$$

The coefficients for $n \neq 0$ are

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{0}(-1)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\pi}(1)e^{-inx}dx \\ &= \left[\frac{e^{-inx}}{2i\pi n}\right]_{-\pi}^{0} + \left[\frac{e^{-inx}}{-2i\pi n}\right]_{0}^{\pi} = \frac{1 - e^{in\pi} + 1 - e^{-in\pi}}{2i\pi n} = \frac{1-(-1)^n}{i\pi n} \\ &= \begin{cases} \frac{2}{i\pi n} & \text{for } n \text{ odd}, \\ 0 & \text{for } n \text{ even}. \end{cases} \end{aligned}$$

The complex Fourier series can be written as

$$f(x) = \sum_{n=0}^{\infty} \frac{2e^{i(2n+1)x}}{i\pi(2n+1)} + \sum_{n=0}^{\infty} \frac{2e^{-i(2n+1)x}}{-\pi(2n+1)}.$$

(b) By evaluating $f(\pi/2)$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Answer: Substituting $\pi/2$ into the Fourier series gives

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \frac{2e^{i(2n+1)\pi/2}}{i\pi(2n+1)} + \sum_{n=0}^{\infty} \frac{2e^{-i(2n+1)\pi/2}}{-i\pi(2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{2i(-1)^n}{i\pi(2n+1)} + \sum_{n=0}^{\infty} \frac{2(-i)(-1)^n}{-i\pi(2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{\pi(2n+1)} + \sum_{n=0}^{\infty} \frac{2(-1)^n}{\pi(2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{4(-1)^n}{\pi(2n+1)} \end{aligned}$$

and hence

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

5. (a) For two functions f and g that have finite integrals, show that the convolution $f * g$ satisfies

$$\int_{-\infty}^{\infty} (f*g)(x)dx = \left(\int_{-\infty}^{\infty} f(x)dx\right)\left(\int_{-\infty}^{\infty} g(x)dx\right).$$

Answer: By making a substitution $\xi = y + x$, it can be seen that

$$\begin{aligned}\int_{-\infty}^{\infty} (f*g)(x)dx &= \int_{-\infty}^{\infty} dx\left(\int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi\right) = \int_{-\infty}^{\infty} dx\int_{-\infty}^{\infty} dy f(x)g(y) \\ &= \left(\int_{-\infty}^{\infty} f(x)dx\right)\left(\int_{-\infty}^{\infty} g(x)dx\right).\end{aligned}$$

(b) Consider the functions

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch f. Calculate $f * g$ and sketch it. Verify that the result from part (a) holds.

Answer: The function f is plotted in Fig. 2. To calculate the convolution, first note that $g(x-\xi) = 1$ if and only if $0 < x - \xi < 1$, which is equivalent to $x - 1 < \xi < x$, and hence

$$(f*g)(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi = \int_{x-1}^{x} f(\xi)d\xi.$$

If $0 < x < 1$ then this becomes

$$(f*g)(x) = \int_{0}^{x} f(\xi)d\xi = \int_{0}^{x} \xi\, d\xi = \frac{x^2}{2}$$

and if $1 \leq x < 2$ then this becomes

$$(f*g)(x) = \int_{x-1}^{1} f(\xi)d\xi = \int_{x-1}^{1} \xi\, d\xi = \frac{1-(x-1)^2}{2} = \frac{x(2-x)}{2}.$$

Otherwise, g will be zero over the entire integration range and hence $(f*g)(x) = 0$. To verify the identity, note that

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{1} x\, dx = \frac{1}{2}, \qquad \int_{-\infty}^{\infty} g(x)dx = \int_{0}^{1} dx = 1.$$

The integral of the convolution is

$$\int_{-\infty}^{\infty} (f*g)(x)dx = \int_{0}^{1} \frac{x^2}{2}dx + \int_{1}^{2} \frac{2x-x^2}{2}dx = \frac{1}{6} + \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_1^2 = \frac{1}{2}$$

and thus the result from part (a) is satisfied.

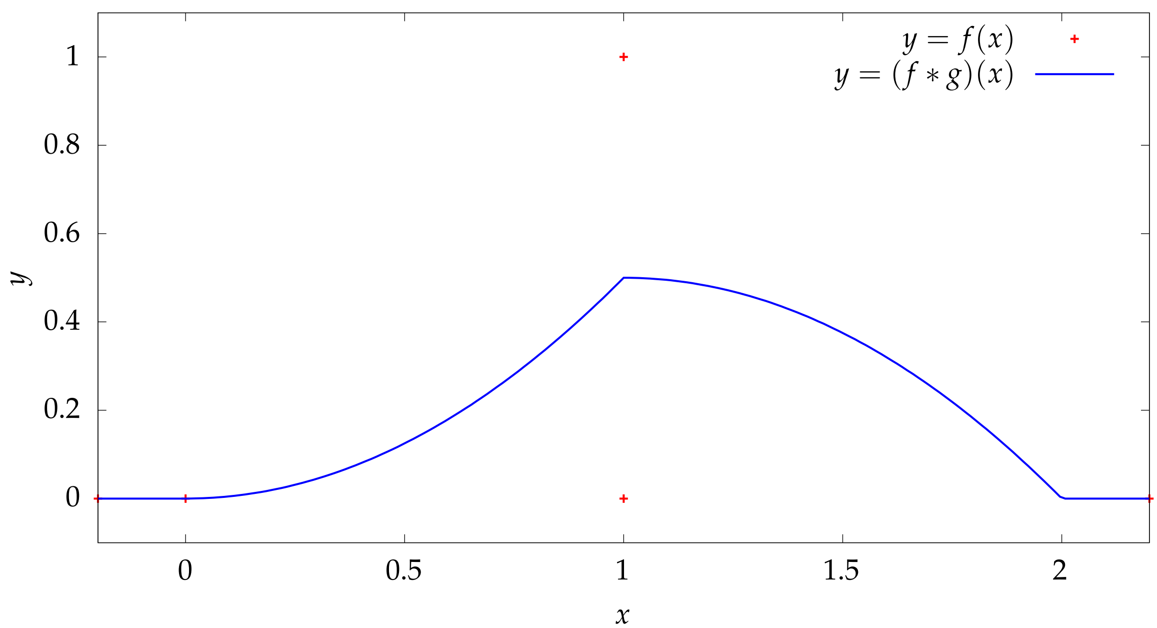

Figure 2: Plot of the function f and a convolution $f * g$ considered in question 5.

6. (a) Show that $y(x) = e^{-x}$ and $y(x) = x$ are solutions to the differential equation

$$(x+1)y'' + xy' - y = 0.$$

Answer: If $y(x) = x$, then $y'(x) = 1$ and $y''(x) = 0$, and hence

$$(x+1)y'' + xy' - y = 0 + x - x = 0$$

so it is a solution. If $y(x) = e^{-x}$, then $y'(x) = -e^{-x}$ and $y''(x) = e^{-x}$, and hence

$$(x+1)y'' + xy' - y = (x+1)e^{-x} - xe^{-x} - e^{-x} = 0$$

so it is a solution.

(b) Consider the differential equation

$$y'' + \frac{xy'}{x+1} - \frac{y}{x+1} = f(x)$$

for $x \geq 0$ subject to the boundary conditions $y(0) = 0$ and $\lim_{x\to\infty} y(x) = 0$. Calculate a Green function solution of the form

$$y(x) = \int_0^\infty G(x,x')f(x')dx'.$$

Answer: To search for a Green function solution, consider solving the equation for $f(x) = \delta(x - x')$ where x' is a parameter. For the two regions $x < x'$ and $x > x'$, the equation is the same as in part (a), and hence the solution will have the form

$$y(x) = \begin{cases} Ax + Be^{-x} & \text{for } x < x', \\ Cx + De^{-x} & \text{for } x \geq x'. \end{cases}$$

Since $y(0) = 0$ it follows that $B = 0$, and since $\lim_{x\to\infty} y(x) = 0$ it follows that $C = 0$. For continuity at $x = x'$,

$$Ax' = De^{-x'}$$

and for a jump in the derivative of 1,

$$1 = y'_+(x') - y'_-(x') = -De^{-x'} - A.$$

These two equations can be solved with

$$A = \frac{-1}{x'+1}, \qquad D = -\frac{e^{x'}x'}{x'+1}$$

and hence the Green function is

$$G(x,x') = \begin{cases} -\frac{x}{x'+1} & \text{for } x < x', \\ -\frac{x'}{x'+1}e^{-(x-x')} & \text{for } x \geq x'. \end{cases}$$

7. (a) Calculate the Laplace transform $F(p)$ of the function

$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1, \\ 0 & \text{for } t \geq 1. \end{cases}$$

Answer: The Laplace transform is given by

$$F(p) = \int_0^\infty f(t)e^{-pt}dp = \int_0^1 e^{-pt}dp = \left[\frac{e^{-pt}}{-p}\right]_0^1 = \frac{1-e^{-p}}{p}.$$

(b) Let $g(t)$ satisfy the differential equation

$$\frac{dg}{dt} + \lambda g = f(t)$$

where $g(0) = 0$ and $\lambda > 0$. Calculate an expression for the Laplace transform of the solution, $G(p)$.

Answer: The Laplace transform of $g'(t)$ is $pG(p) - p(0)$ and hence

$$(pG(p) - p(0)) + \lambda G(p) = \frac{1-e^{-p}}{p},$$

which can be rearranged to give

$$G(p) = \frac{1-e^{-p}}{p(p+\lambda)}.$$

(c) Use the Bromwich inversion integral

$$g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}G(p)dp$$

to calculate $g(t)$, where c is a positive constant.

Answer: The integral can be written as

$$g(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{pt}-e^{p(t-1)}}{p(p+\lambda)}$$

and has two simple poles at $p = 0, -\lambda$. If $0 \leq t < 1$, then the $e^{p(t-1)}$ term will be small as $\text{Re}(p) \to \infty$. Closing this term with a large semicircle in the right half plane will enclose no poles. Hence the only contributions come from the e^{pt} term:

$$\begin{aligned} g(x) &= \text{Res}\left(\frac{e^{pt}}{p(p+\lambda)}, p=0\right) + \text{Res}\left(\frac{e^{pt}}{p(p+\lambda)}, p=-\lambda\right) \\ &= \lim_{p\to 0}\frac{e^{pt}}{(p+\lambda)} + \lim_{p\to-\lambda}\frac{e^{pt}}{p} = \frac{1-e^{-\lambda t}}{\lambda}. \end{aligned}$$

If $t > 1$ then both exponentials will be small as $\text{Re}(p) \to -\infty$ in the left half plane, and hence

$$\begin{aligned} g(x) &= \text{Res}\left(\frac{e^{pt}-e^{p(t-1)}}{p(p+\lambda)}, p=0\right) + \text{Res}\left(\frac{e^{pt}-e^{p(t-1)}}{p(p+\lambda)}, p=-\lambda\right) \\ &= \lim_{p\to 0}\frac{e^{pt}-e^{p(t-1)}}{(p+\lambda)} + \lim_{p\to-\lambda}\frac{e^{pt}-e^{p(t-1)}}{p} = \frac{-e^{-\lambda t}+e^{-\lambda(t-1)}}{\lambda} = \frac{(e^{\lambda}-1)e^{-\lambda t}}{\lambda}. \end{aligned}$$

8. (a) If $z = x + iy$ is a complex number and $|z| = r$, show explicitly that $\bar{z} = r^2/z$.

Answer: The complex conjugate satisfies the relation

$$\bar{z} = x - iy = \frac{(x - iy)(x + iy)}{x + iy} = \frac{x^2 + y^2}{x + iy} = \frac{r^2}{z}.$$

(b) Consider the contour integral

$$I(r) = \oint_{C(r)} \frac{dz}{(\bar{z} + 1)(z + 2)}$$

where $C(r)$ is a circle of radius r centered at 0. By using residue calculus or otherwise, evaluate $I(r)$ for the three cases of (i) $r > 2$, (ii) $1 < r < 2$, and (iii) $0 < r < 1$.

Answer: To use residue calculus, the integrand must be analytic. By making use of the relation from part (a), and the fact that $|z| = r$ on the contour,

$$I(r) = \oint_{C(r)} \frac{dz}{\left(\frac{r^2}{z} + 1\right)(z + 2)} = \oint_{C(r)} \frac{z\,dz}{(r^2 + z)(z + 2)}.$$

The integrand has simple poles at $z = -r^2$ and $z = -2$; the residues at these points are

$$\text{Res}\left(\frac{z\,dz}{(r^2 + z)(z + 2)}, z = -2\right) = \lim_{z \to -2} \frac{z}{r^2 + z} = \frac{2}{2 - r^2}$$

and

$$\text{Res}\left(\frac{z\,dz}{(r^2 + z)(z + 2)}, z = -r^2\right) = \lim_{z \to -r^2} \frac{z}{z + 2} = \frac{r^2}{r^2 - 2}.$$

If $r > 2$ then $C(r)$ will only enclose the pole at $z = -2$, and hence

$$I(r) = 2\pi i \frac{2}{2 - r^2} = \frac{4\pi i}{2 - r^2}.$$

If $1 < r < 2$ then $C(r)$ will not enclose any singularities and hence $I(r) = 0$. If $0 < r < 1$ then $C(r)$ will enclose the pole at $z = -r^2$ and hence

$$I(r) = 2\pi i \frac{r^2}{r^2 - 2} = \frac{2\pi i r^2}{r^2 - 2}.$$

9. The displacement $x(t)$ of a mass on the end of a damped spring undergoes vibrations of the form

$$\ddot{x} + 2\mu\dot{x} + kx = 0$$

where $k > 0$, and the damping coefficient μ satisfies $0 < \mu < \sqrt{k}$.

(a) Solve the equation for initial conditions $\dot{x}(0) = 0$, and $x(0) = a$. *(Note: it may help to define $q = \sqrt{k - \mu^2}$ and express your answer in terms of q.)*

Answer: Searching for solutions of the form e^{bx} gives $b^2 + 2\mu b + k = 0$ and thus $b = -\mu \pm \sqrt{m^2 - k} = -\mu \pm iq$. Hence the general solution is

$$x(t) = e^{-\mu t}(A \cos qt + B \sin qt)$$

Since $x(0) = a$, it follows that $A = a$. The derivative is

$$\dot{x}(t) = -\mu e^{-\mu t}(a \cos qt + B \sin qt) + e^{-\mu t}(-aq \sin qt + Bq \cos qt)$$

and $\dot{x}(0) = 0$ implies $B = a\mu/q$. Hence

$$x(t) = ae^{-\mu t}\left(\cos qt + \frac{\mu \sin qt}{q}\right).$$

(b) Let $t_0 = 0$, and define t_j to be the sequence of successive times when the mass is stationary, so that $\dot{x}(t_j) = 0$. Calculate an expression for t_j. Sketch the curve $x(t)$ over the range $t_0 \le t \le t_3$ and indicate t_1, t_2, t_3 on your sketch.

Answer: The derivative is

$$\dot{x}(t) = -ae^{-\mu t}\left(\frac{\mu^2}{q} + q\right)\sin qt.$$

This will be zero when $\sin qt$ is zero, and hence $t_j = \frac{j\pi}{q}$. A plot of $x(t)$ is shown in Fig. 3.

(c) For each j, define $x_j = x(t_j)$. By considering $|x_{j+1} - x_j|$, calculate the total absolute distance that the mass covers as it comes to rest.

Answer: It can be seen that

$$x_j = x(t_j) = ae^{-\mu j\pi/q}\left(\cos\frac{qj\pi}{q} + \frac{\mu}{q}\sin\frac{qj\pi}{q}\right) = ae^{-\mu\pi j/q}\left((-1)^j + 0\right) = ae^{-\mu\pi j/q}.$$

The distance that the mass travels between t_j and t_{j+1} is

$$|x_{j+1} - x_j| = |ae^{-\mu\pi(j+1)/q} + ae^{-\mu\pi j/q}| = ae^{-\mu\pi j/q}(e^{-\mu\pi/q} + 1)$$

and hence the total distance covered is a geometric series,

$$\sum_{j=0}^{\infty}|x_{j+1} - x_j| = a(e^{-\mu\pi/q} + 1)\sum_{j=0}^{\infty} e^{-\mu\pi j/q} = \frac{a(1 + e^{-\mu\pi/q})}{1 - e^{-\mu\pi/q}} = \frac{a}{\tanh\frac{\mu\pi}{2q}}.$$

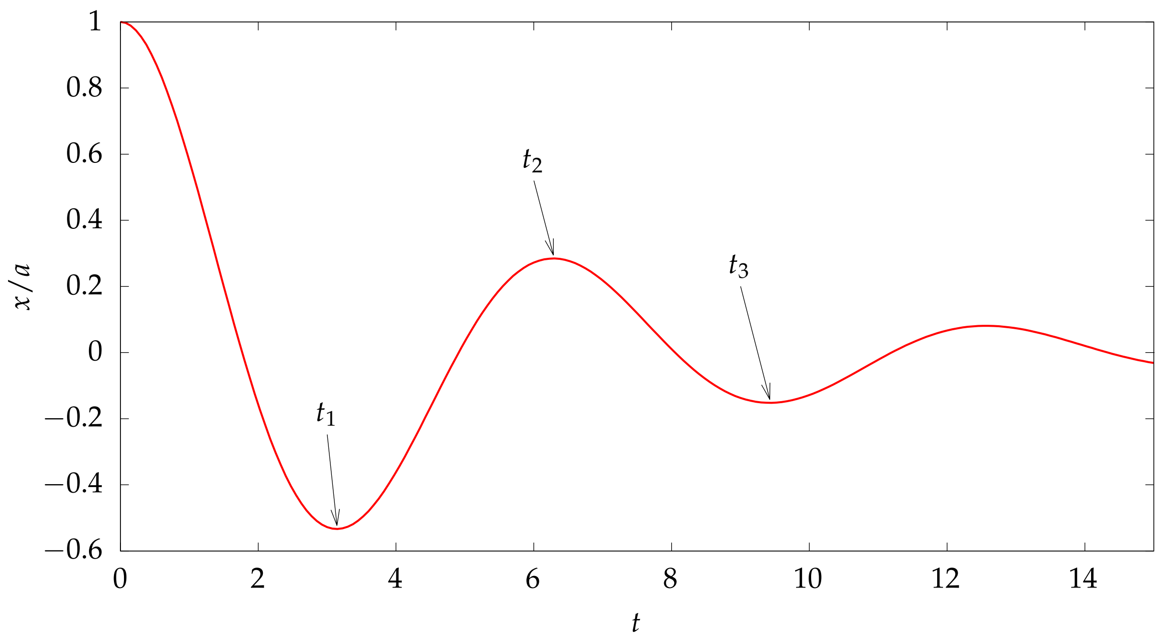

Figure 3: Plot of the mass position, with the times t_1, t_2, and t_3 marked on, where the mass becomes stationary. Parameters of $\mu = 1/5$ and $q = 1$ are used.

10. Consider a curve described by $y(x)$ from $(x, y) = (-1, 0)$ to $(x, y) = (1, 0)$. Find the shape of the curve that maximizes the area underneath it, given by

$$\int_{-1}^{1} y \, dx,$$

subject to the constraint that the total arc length of the curve is L. Find the explicit form of $y(x)$ for the cases of $L = \pi$ and $L = \pi/\sqrt{2}$.

Answer: To minimize subject to a constraint, a Lagrange multiplier λ can be introduced.

$$\int_{-1}^{1} F(x, y, y', \lambda) dx = \int_{-1}^{1} \left(y + \lambda\sqrt{1 + y'^2} \right) dx$$

Since there is no dependence on x, the Beltrami identity can be used and

$$C = F - y' \frac{\partial F}{\partial y'} = y + \lambda\sqrt{1 + y'^2} - \frac{\lambda y'^2}{\sqrt{1 + y'^2}} = y + \frac{\lambda}{\sqrt{1 + y'^2}}$$

for some constant C. Hence

$$(C - y)^2 = \frac{\lambda^2}{1 + y'^2},$$

which can be rearranged to give

$$y'^2 = \frac{\lambda^2 - (C-y)^2}{(C-y)^2}.$$

Therefore

$$\int \frac{(C-y)dy}{\sqrt{\lambda^2 - (C-y)^2}} = \int dx \qquad \Longrightarrow \qquad x - x_0 = \sqrt{\lambda^2 - (C-y)^2}.$$

Hence

$$(y-C)^2 + (x-x_0)^2 = \lambda^2,$$

which is the equation of a circle of radius λ centered on (x_0, C). To find the explicit form of the solutions for the two values of L, it easiest to examine the problem geometrically. By symmetry $x_0 = 0$. For a given value of λ, it follows that $C = \sqrt{\lambda^2 - 1}$ in order for $y(\pm 1) = 0$. Using geometry, the angle of the circle between $(-1, 0)$ and $(1, 0)$ is

$$\theta = 2\cos^{-1}\frac{\sqrt{\lambda^2 - 1}}{\lambda}$$

and hence the total length is

$$L = \lambda\theta = 2\lambda\cos^{-1}\frac{\sqrt{\lambda^2 - 1}}{\lambda}.$$

The two values of L that are given correspond to simple choices of λ. If $\lambda = 1$ then $C = 0$ and $L = 2\cos^{-1} 0 = 2\frac{\pi}{2} = \pi$ as required. If $\lambda = \sqrt{2}$ then $C = 1$ and $L = 2\sqrt{2}\cos^{-1} 1/\sqrt{2} = 2\sqrt{2}\frac{\pi}{4} = \pi/\sqrt{2}$ as required. Explicitly, the solutions are

$$y(x) = \sqrt{1 - x^2}, \qquad y(x) = -1 + \sqrt{2 - x^2}$$

respectively. Plots of these functions are shown in Fig. 4.

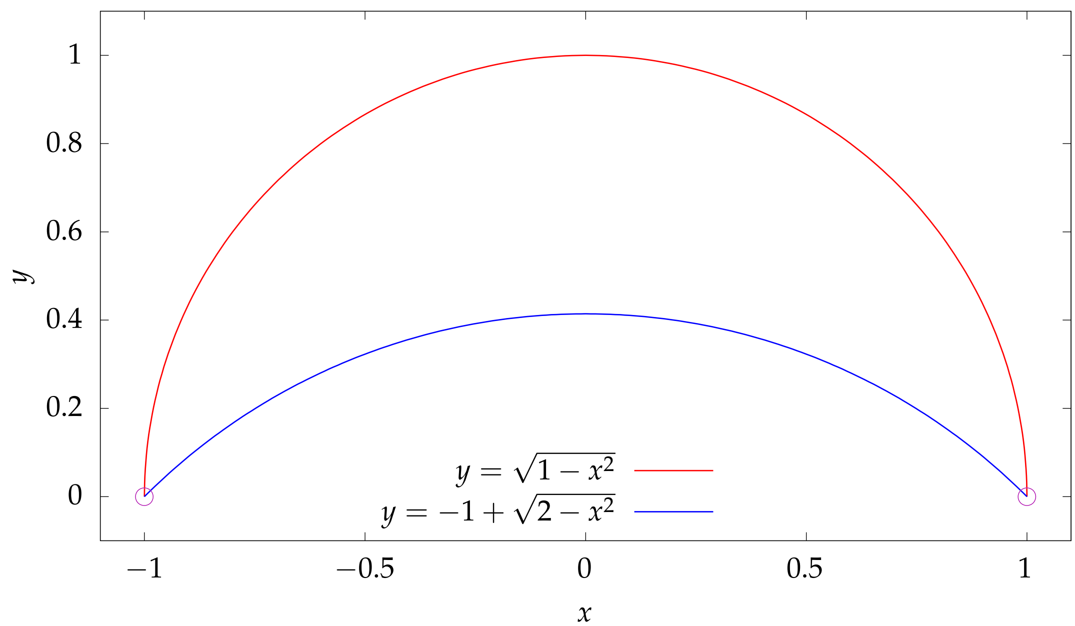

Figure 4: Plots of the functions from $(-1, 0)$ to $(1, 0)$ that maximize the area under the curve, subject to the constraints that the arc lengths are π and $\pi/\sqrt{2}$.