Math 121A: Final exam

1. (a) Calculate the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

   (b) Solve the linear system

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}.$$

2. (a) Calculate the radius of convergence R of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n(-3)^n}.$$

   By considering the series for $x = \pm R$ and using appropriate series tests, determine the exact range of x for which it converges.

   (b) For the function $f(t) = -5 + 2t^2$, calculate $f(0)$, $f(1)$, $f(2)$, and $f(3)$. Use the results to sketch f over the range $-3 \leq t \leq 3$.

   (c) Determine the precise set of values of t for which the series

$$\sum_{n=1}^{\infty} \frac{[f(t)]^n}{n(-3)^n}$$

   will converge.

3. (a) Let a function $f(x)$ have the Fourier transform $\tilde{f}(\alpha)$. Let $g(x) = f(-x)$ and $h(x) = xf(x)$. Show that the Fourier transforms of g and h are given by $\tilde{g}(\alpha) = \tilde{f}(-\alpha)$ and $\tilde{h}(\alpha) = i\tilde{f}'(\alpha)$.

   (b) Determine the Fourier transform $\tilde{f}(\alpha)$ of the function

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

   (c) By using the above results, or otherwise, determine the Fourier transform $\tilde{s}(\alpha)$ of

$$s(x) = x^2 e^{-|x|}.$$

   Show that $\tilde{s}(\alpha)$ is real.

4. (a) Calculate the complex Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for the function

$$f(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0 \end{cases}$$

defined on the interval $-\pi \leq x < \pi$.

(b) By evaluating $f(\pi/2)$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

5. (a) For two functions f and g that have finite integrals, show that the convolution $f * g$ satisfies

$$\int_{-\infty}^{\infty} (f * g)(x) dx = \left(\int_{-\infty}^{\infty} f(x) dx \right) \left(\int_{-\infty}^{\infty} g(x) dx \right).$$

(b) Consider the functions

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Sketch f. Calculate $f * g$ and sketch it. Verify that the result from part (a) holds.

6. (a) Show that $y(x) = e^{-x}$ and $y(x) = x$ are solutions to the differential equation

$$(x+1)y'' + xy' - y = 0.$$

(b) Consider the differential equation

$$y'' + \frac{xy'}{x+1} - \frac{y}{x+1} = f(x)$$

for $x \geq 0$ subject to the boundary conditions $y(0) = 0$ and $\lim_{x \to \infty} y(x) = 0$. Calculate a Green function solution of the form

$$y(x) = \int_0^{\infty} G(x, x') f(x') dx'.$$

7. (a) Calculate the Laplace transform $F(p)$ of the function

$$f(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1, \\ 0 & \text{for } t \geq 1. \end{cases}$$

(b) Let $g(t)$ satisfy the differential equation

$$\frac{dg}{dt} + \lambda g = f(t)$$

where $g(0) = 0$ and $\lambda > 0$. Calculate an expression for the Laplace transform of the solution, $G(p)$.

(c) Use the Bromwich inversion integral

$$g(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} G(p) dp$$

to calculate $g(t)$, where c is a positive constant.

8. (a) If $z = x + iy$ is a complex number and $|z| = r$, show explicitly that $\bar{z} = r^2/z$.

   (b) Consider the contour integral

$$I(r) = \oint_{C(r)} \frac{dz}{(\bar{z}+1)(z+2)}$$

where $C(r)$ is a circle of radius r centered at 0. By using residue calculus or otherwise, evaluate $I(r)$ for the three cases of (i) $r > 2$, (ii) $1 < r < 2$, and (iii) $0 < r < 1$.

9. The displacement $x(t)$ of a mass on the end of a damped spring undergoes vibrations of the form

$$\ddot{x} + 2\mu\dot{x} + kx = 0$$

where $k > 0$, and the damping coefficient μ satisfies $0 < \mu < \sqrt{k}$.

   (a) Solve the equation for initial conditions $\dot{x}(0) = 0$, and $x(0) = a$. *(Note: it may help to define $q = \sqrt{k - \mu^2}$ and express your answer in terms of q.)*

   (b) Let $t_0 = 0$, and define t_j to be the sequence of successive times when the mass is stationary, so that $\dot{x}(t_j) = 0$. Calculate an expression for t_j. Sketch the curve $x(t)$ over the range $t_0 \leq t \leq t_3$ and indicate t_1, t_2, t_3 on your sketch.

   (c) For each j, define $x_j = x(t_j)$. By considering $|x_{j+1} - x_j|$, calculate the total absolute distance that the mass covers as it comes to rest.

10. Consider a curve described by $y(x)$ from $(x, y) = (-1, 0)$ to $(x, y) = (1, 0)$. Find the shape of the curve that maximizes the area underneath it, given by

$$\int_{-1}^{1} y \, dx,$$

subject to the constraint that the total arc length of the curve is L. Find the explicit form of $y(x)$ for the cases of $L = \pi$ and $L = \pi/\sqrt{2}$.