# Math 104: Homework 9 solutions

1. (a) For $x \neq 0$,

$$\begin{aligned} f'(x) &= x^2 \frac{-1}{x^2} \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \\ &= \cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) \end{aligned}$$

where the product rule and chain rule have been employed.

(b) At $x = 0$,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$.

(c) Consider the sequence $s_n = 1/(2n\pi)$. Then

$$f'(s_n) = \cos(2n\pi) + \frac{1}{n\pi} \sin(2n\pi) = 1$$

and hence $\lim_{n \to \infty} f'(s_n) = 1$. However, $s_n \to 0$ but $f'(0) = 0$. Hence $f'$ is not continuous at 0.

2. (a) Consider $\epsilon > 0$; since $f_n$ is uniformly Cauchy, then there exists an $N$ such that $n, m > N$ implies

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

for all $x \in [a, b]$, so that

$$-\frac{\epsilon}{2} < f_n(x) - f_m(x) < \frac{\epsilon}{2}.$$

Taking the limit as $x \to x_0$ implies that

$$-\frac{\epsilon}{2} \leq l_n - l_m \leq \frac{\epsilon}{2}.$$

This result was proved in detail on an earlier homework (Ross Exercise 20.16). Hence

$$|l_n - l_m| \leq \frac{\epsilon}{2} < \epsilon$$

and $l_n$ is a Cauchy sequence. Hence it converges to a limit $l$.

(b) Choose $\epsilon > 0$. Then since $f_n$ converges uniformly to $f$, there exists an $N_1$ such that $n > N_1$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in [a,b]$. Since $l_n \to l$, there exists an $N_2$ such that

$$|l_n - l| < \frac{\epsilon}{3}.$$

Define $N = \max\{N_1, N_2\} + 1$, and consider $f_N$. Since $\lim_{x \to x_0} f_N(x) = l_N$, there exists a $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies that

$$|f_N(x) - l_N| < \frac{\epsilon}{3}.$$

Thus for $0 < |x - x_0| < \delta$,

$$|f(x) - l| \leq |f(x) - f_N(x)| + |f_N(x) - l_N| + |l_N - l| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

and hence $\lim_{x \to x_0} f(x) = l$.

3. First suppose that $h$ is differentiable. If $h$ is differentiable, then it is also continuous. Consider the sequence $s_n = a + \lambda n^{-1}$ where $\lambda > 0$ is chosen so that the $s_n$ lie within $I$ for all $n \in \mathbb{N}$. Then $h(s_n) = g(s_n)$ and since $g$ is continuous, $\lim_{n \to \infty} g(s_n) = g(a)$. Similarly, consider $t_n = a - \mu n^{-1}$ where $\mu > 0$ ensures $t_n \in I$ for all $n \in \mathbb{N}$. Then $h(t_n) = f(t_n)$, and $\lim_{n \to \infty} f(t_n) = f(a)$. Since continuity implies that $\lim_{n \to \infty} h(s_n) = \lim_{n \to \infty} h(t_n)$, then $f(a) = g(a)$. Recall from Theorem 20.10 that $\lim_{x \to a}$ exists if and only if the negative and positive limits are the same. The positive limit is

$$h'(a) = \lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a).$$

By making use of the fact that $f(a) = g(a)$, the negative limit is

$$h'(a) = \lim_{x \to a^-} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Hence $f'(a) = g'(a)$.

Now suppose the converse, that $f(a) = g(a)$ and $f'(a) = g'(a)$. Then the positive limit is

$$\lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a)$$

and the negative limit is

$$\lim_{x \to a^-} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Thus, since both the positive and negative limits exist and are equal, then by Theorem 20.10 the limit as $x \to a$ exists and is equal to $f'(a) = g'(a)$. Hence $h$ is differentiable at $a$.

4. First suppose that $s_n = s_{n-1}$ for some $n \in \mathbb{N}$. Then $f(s_n) = s_n$, and thus $s_m = s_n$ for all $m \geq n-1$, so the sequence becomes constant and is therefore convergent.

   Now assume that $s_n \neq s_{n-1}$ for all $n \in \mathbb{N}$. Applying the Mean Value Theorem to the interval from $s_n$ to $s_{n-1}$ shows that there is a value $y$ between $s_n$ and $s_{n-1}$ such that
$$\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} = f'(y)$$
   Hence
$$(s_{n+1} - s_n) = (s_n - s_{n-1})f'(y)$$
   and since $|f'(y)| \leq a$ it follows that
$$|s_{n+1} - s_n| \leq a|s_n - s_{n-1}|.$$
   Applying this result repeatedly gives
$$|s_{n+1} - s_n| \leq a^n|s_1 - s_0|.$$
   Now, for any $m$ and $n$ where $n > m$, the triangle inequality gives
$$|s_n - s_m| \leq \sum_{k=m}^{n-1} |s_{k+1} - s_k| \leq \sum_{k=m}^{n-1} a^k|s_1 - s_0| < \frac{|s_1 - s_0|a^m}{1-a}.$$
   Consider any $\epsilon > 0$: since $\lim_{k\to\infty} a^k = 0$, there exists an $N$ such that $k > N$ implies that $|s_1 - s_0|a^m/(1-a) < \epsilon$, and hence $m, n > N$ implies that
$$|s_n - s_m| < \epsilon$$
   so $(s_n)$ is a Cauchy sequence, and hence convergent.

5. (a) If $f''(x) \geq 0$ for $x > 0$, then $f'(x)$ is an increasing function for $x \geq 0$. Suppose $y > x$. Then by the mean value theorem, there exists a $c \in (0, x)$ such that
$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$
   and there exists and $d \in (x, y)$ such that
$$\frac{f(y) - f(x)}{y - x} = f'(d).$$
   Since $d > c$, then $f'(d) \geq f'(c)$, and hence
$$\frac{f(x)}{x} \leq \frac{f(y) - f(x)}{y - x}$$

from which it follows that

$$yf(x) - xf(x) \leq xf(y) - xf(x)$$

so

$$yf(x) \leq xf(y)$$

and hence

$$\frac{f(x)}{x} = \frac{f(y)}{y}.$$

Since this is true for any $x, y$ where $x < y$, then $f(x)/x$ is increasing function.

(b) Consider $y > x$. By following the same steps as above, there exists a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

and there exists and $d \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(d)$$

Since $f(x)/x$ is increasing, then $f(x)/x \leq f(y)/y$, and hence

$$yf(x) - xf(x) \leq xf(y) - xf(x)$$

so

$$\frac{f(x)}{x} \leq \frac{f(y) - f(x)}{y - x}.$$

Therefore $f'(c) \leq f'(d)$. By applying the mean value theorem to $f'$, there exists an $e \in (c, d)$ such that

$$f''(e) = \frac{f'(d) - f'(c)}{d - c} \geq 0.$$

To show that it is not necessarily the case that $f''(x) \geq 0$ for all $x > 0$, consider the function $f(x) = x(1 - e^{-x})$. Then $f(0) = 0$ and

$$\frac{f(x)}{x} = 1 - e^{-x}$$

so $f$ satisfies the conditions described in the exercise. Consider two values $x, y$ where $x < y$. Then $e^{-x} > e^{-y}$, so $1 - e^{-x} < 1 - e^{-y}$, and hence $f(x)/x < f(y)/y$, so $f(x)/x$ is an increasing function. The first derivative is

$$f'(x) = 1 - e^{-x} + xe^{-x}$$

and the second derivative is

$$f''(x) = e^{-x} + e^{-x} - xe^{-x} = (2 - x)e^{-x}.$$

Hence $f''(3) = -e^{-3} < 0$, so it is not the case that $f''(x) \geq 0$ for all $x > 0$.

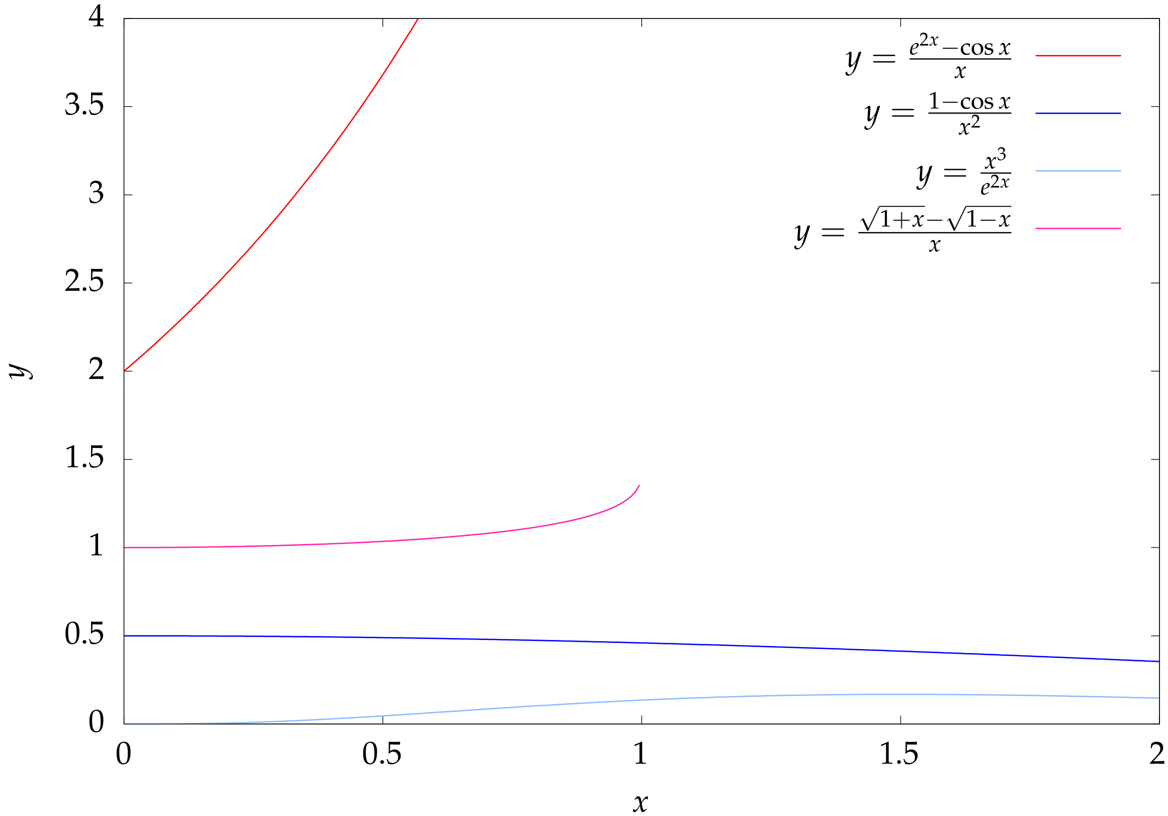


Figure 1: Graphs for the question on computing limits using L'Hôpital's rule.

6. Graphs of the four functions in the question are shown in Fig. 1, confirming the limits for the cases where $x \to 0$.

   (a) Let $f(x) = e^{2x} - \cos x$ and $g(x) = x$. Then $f'(x) = 2e^{2x} + \sin x$ and $g'(x) = 1$. Then
   $$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} 2e^{2x} + \sin x = \lim_{x\to 0} 2.$$
   Since this limit exists and both $f(x)$ and $g(x)$ have limit zero as $x \to 0$, L'Hôpital's rule can be applied, and thus
   $$\lim_{x\to 0} \frac{e^{2x} - \cos x}{x} = 2.$$

   (b) Let $f(x) = 1 - \cos x$ and $g(x) = x^2$. To evaluate the limit $f(x)/g(x)$, L'Hôpital's rule should be applied twice. Taking first derivatives gives $f'(x) = \sin x$ and $g'(x) = 2x$, both of which have limit zero as $x \to 0$. Taking second derivatives gives $f''(x) = \cos(x)$ and $g''(x) = 2$, and thus
   $$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}.$$
   Hence
   $$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}.$$

   (c) L'Hôpital's rule can be applied several times to show that
   $$\begin{aligned}\lim_{x\to\infty} \frac{x^3}{e^{2x}} &= \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} \\ &= \lim_{x\to\infty} \frac{6x}{4e^{2x}} \\ &= \lim_{x\to\infty} \frac{6}{8e^{2x}} \\ &= \lim_{x\to\infty} \frac{3}{4} e^{-2x} = 0.\end{aligned}$$

   (d) Let $f(x) = \sqrt{1+x} - \sqrt{1-x}$ and $g(x) = x$. Both $f$ and $g$ have limit zero as $x \to 0$. The first derivatives are
   $$f'(x) = \frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}}$$
   and $g(x) = 1$. Hence
   $$\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{\frac{1}{2} + \frac{1}{2}}{1} = 1.$$

7. (a) The definition of derivative is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

and thus by writing $h = x - a$, this can be rewritten as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

If $h$ is negated, then

$$\begin{aligned} f'(a) &= \lim_{-h\to 0} \frac{f(a-h)-f(a)}{-h} \\ &= \lim_{h\to 0} \frac{f(a)-f(a-h)}{h}. \end{aligned}$$

Now consider $L_1$; this can be rewritten as

$$\begin{aligned} \lim_{h\to 0} L_1(a,h) &= \lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} \\ &= \lim_{h\to 0} \left( \frac{f(a+h)-f(a)}{2h} + \frac{f(a)-f(a-h)}{2h} \right) \end{aligned}$$

and since both of the terms have a well-defined limit and limits are additive,

$$\lim_{h\to 0} L_1(a,h) = \frac{f'(a)}{2} + \frac{f'(a)}{2} = f'(a).$$

Since $L_2$ can be written as

$$L_2(a,h) = \frac{8L_1(a,h)}{6} - \frac{L_1(a,2h)}{3}$$

it follows that

$$\lim_{h\to 0} L_2(a,h) = \frac{8f'(a)}{6} - \frac{f'(a)}{3} = f'(a).$$

(b) If $f(x) = x^5$, then $f'(a) = 5a^4$. Since

$$2hL_1(a,h) = (a+h)^5 - (a-h)^5 = 10a^4h + 20a^2h^3 + 2h^5$$

then

$$|L_1(a,h) - f'(a)| = |10a^2h^2 + h^5|.$$

Thus, as $h \to 0$, $L_1(a,h)$ converges quadratically to $f'(a)$. Now consider $L_2(a,h)$; by making use of the above result, note that

$$8(a+h)^5 - 8(a-h)^5 = 80a^4h + 160a^2h^3 + 16h^5$$

and

$$-(a+2h)^5 + (a-2h)^5 = -20a^4h - 160a^2h^3 - 64h^5.$$

Thus

$$\begin{aligned} 12hL_2(a,h) &= -(a+2h)^5 + 8(a+h)^5 - 8(a-h)^5 + (a-2h)^5 \\ &= 80a^4h + 160a^2h^3 + 16h^5 - 20a^4h - 160a^2h^3 - 64h^5 \\ &= 60a^4h - 48h^5 \end{aligned}$$

so

$$|L_2(a,h) - f'(a)| = |4h^5|.$$

Hence, as $h \to 0$, $L_2(a,h)$ converges quartically to $f'(a)$. The formulae $L_1$ and $L_2$ for evaluating derivatives are important in computational approaches to simulating partial differential equations. The standard definition of the derivative is "first-order accurate" meaning that as $h \to 0$, the error between the computed derivative and the actual derivative scales like $|h|$. However, the formulae above, which are referred to as centered differences, have errors which scale according to $|h|^2$ and $|h^4|$, and are referred to as second-order and fourth-order accurate respectively.

8. (a) Since $|\cos x| \leq 1$ and $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, it follows that $|\cos x \sin x| \leq 1$ for all $x \in \mathbb{R}$, and hence

$$f(x) \geq x - 1.$$

Since $\lim_{x\to\infty} x = \infty$, it follows that $\lim_{x\to\infty} f(x) = \infty$. To find the limit of $g(x)$, note that $e^{\sin x} \in [e^{-1}, e]$, and thus $g(x) \geq f(x)e^{-1}$, so $\lim_{x\to\infty} g(x) = \infty$.

(b) The derivative of $f$ is

$$f'(x) = 1 + \cos x^2 - \sin^2 x = 2\cos^2 x.$$

The function $g$ can be written as $g(x) = e^{\sin x} f(x)$, and hence

$$g'(x) = e^{\sin x} \cos x f(x) + e^{\sin x} f'(x) = e^{\sin x} \cos x (2\cos x + f(x)).$$

(c) If $x > 3$,

$$f(x) + 2\cos x \geq x - 1 + 2\cos x \geq x - 3 > 0.$$

Hence if $x > 3$ and $\cos(x) \neq 0$, then $g'(x) \neq 0$, so

$$\frac{f'(x)}{g'(x)} = \frac{2\cos^2 x}{e^{\sin x}\cos x(2\cos x + f(x))} = \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)}.$$

(d) For $x > 3$,

$$\left| \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} \right| \leq \left| \frac{2e}{x-3} \right|$$

and since $\lim_{x\to\infty} \frac{1}{|x-3|} = 0$ it follows that

$$\lim_{x\to\infty} \frac{2e^{-\sin x}\cos x}{2\cos x + f(x)} = 0.$$

However, for $x > 1$,

$$\frac{f(x)}{g(x)} = \frac{x + \cos x \sin x}{e^{\sin x}(x + \cos x \sin x)} = e^{-\sin x}.$$

For all $n \in \mathbb{N}$, $e^{-\sin x} = e^{-1}$ if $x = 2n\pi + \pi/2$ and $e^{-\sin x} = e$ if $x = 2n\pi + 3\pi/2$. Hence $\lim_{x\to\infty} \frac{f}{g}$ does not exist.