Math 104: Homework 8 solutions

1. (a) Figure 1(a) shows graphs of $f(x)$ compared to $f(x/2)$, $f(x/3)$, and $f(x/4)$. The graphs have the same shape, but are stretched in the x direction. In general, for any function $g(x)$, the graph of $y = g(\lambda x)$ will be stretched by a factor of λ^{-1} in the x direction.

   (b) Figure 1(b) shows graphs of $f(x)$ compared to $2f(x)$, $f(x+1/2)$, and $f(x) - \frac{1}{2}$. The all correspond to different shifts and scalings of f. The function $2f(x)$ is scaled by a factor of 2 in the y direction. The function $f(x+1/2)$ is shifted by $-1/2$ in the x direction, and the function $f(x) - 1/2$ is shifted by a $-1/2$ in the y direction.

   (c) The function $|f(x) - 1/2|$ is shown in Fig. 2(a). This can be graphically interpreted as reflecting the parts of $f(x) - 1/2$ below the y axis to above it.

   (d) Figure 2(b) shows graphs of $f(x^2)$ and $f(x)^2$, both of which can be interpreted as nonlinear scalings. The graph of $f(x^2)$ is the same as $f(x)$, but with a scaling applied to the x axis. The graph of $f(x)^2$ scales each value of $f(x)$ in the y direction. The two functions agree over the domain $[0,1]$.

2. (a) From the definition,

$$f_1(x) = \frac{x^2}{1+x^2}.$$

   As $x \to \infty$, it becomes much larger than 1, and thus the x^2 terms dominate and $f_1(x) \to 1$. Close to zero, when x^2 is small $f_1(x) \approx x^2$, looking locally like a quadratic. The function $f_1(x)$ is plotted in Fig. 3, where these features are visible.

   (b) It can be seen that

$$f_n(x) = \frac{nx^2}{1+nx^2} = \frac{(\sqrt{n}x)^2}{1+(\sqrt{n}x)^2} = f_1(\sqrt{n}x)$$

   Thus the graphs of f_n have the same shape as f_1, but are scaled by a factor of $1/\sqrt{n}$ in the x direction.

   (c) The limit function f and strip of width $1/4$ are shown in Fig. 3. It can be seen that as n increases, more of the curve f_n lies within the strip. However, since $f_n(0) = 0$, the curves must eventually exit the strip, and there will always be some part outside. Hence f_n does not converge uniformly.

3. (a) The functions f_0, f_1, and f_2 are plotted in Fig. 4. Since the values of f_0 oscillate from -1 to 1 infinitely often in the region close to 0, then f_0 is not continuous here. However f_1 and f_2 are continous. f_0 and f_1 are not differentiable at 0, but the oscilliations in f_2 become small enough near 0 that it is differentiable there.

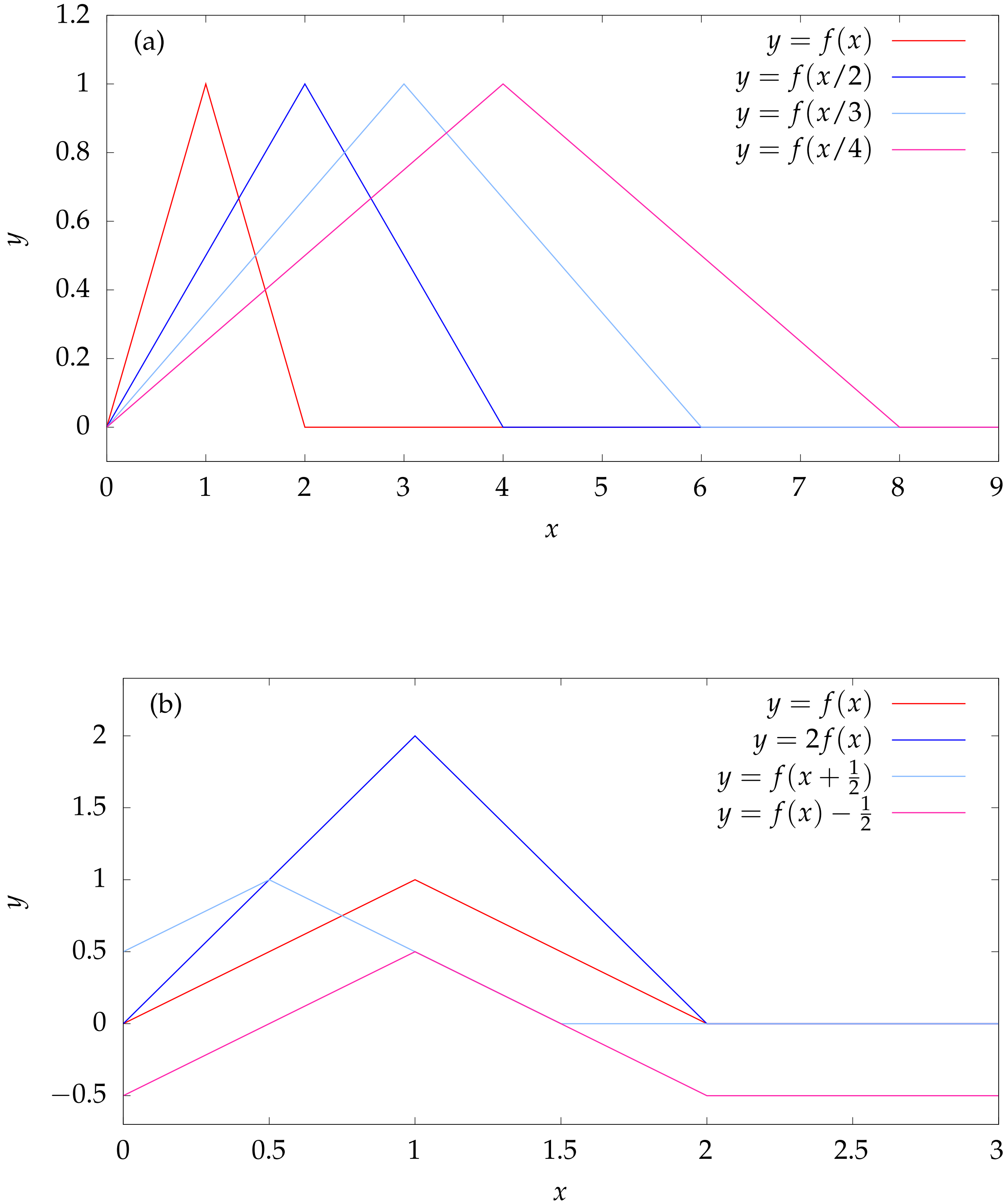

Figure 1: Graphs for the first half of question 1, showing various scalings and translations of $f(x)$.

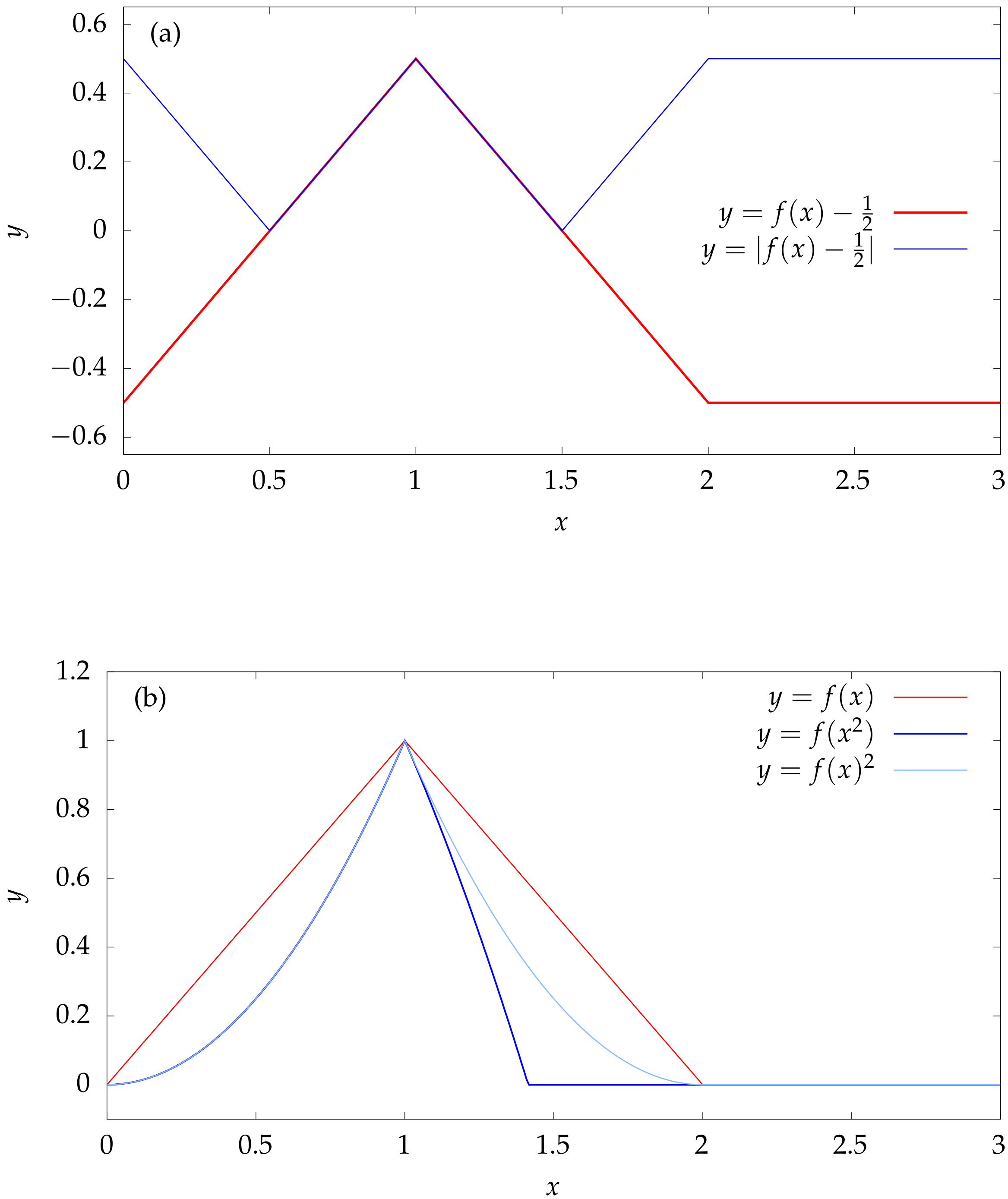

Figure 2: Graphs for the second half of question 1, showing the effect of an absolute value, and performing a nonlinear scaling.

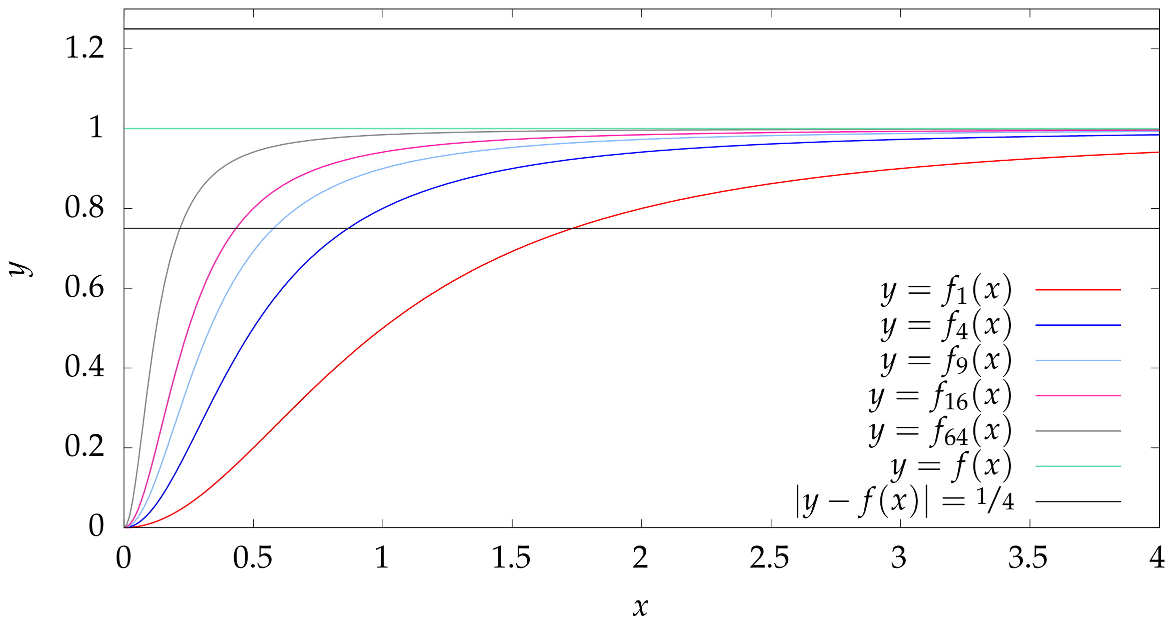

Figure 3: Graph for question 2, showing a sequence of functions with a pointwise but not uniform limit.

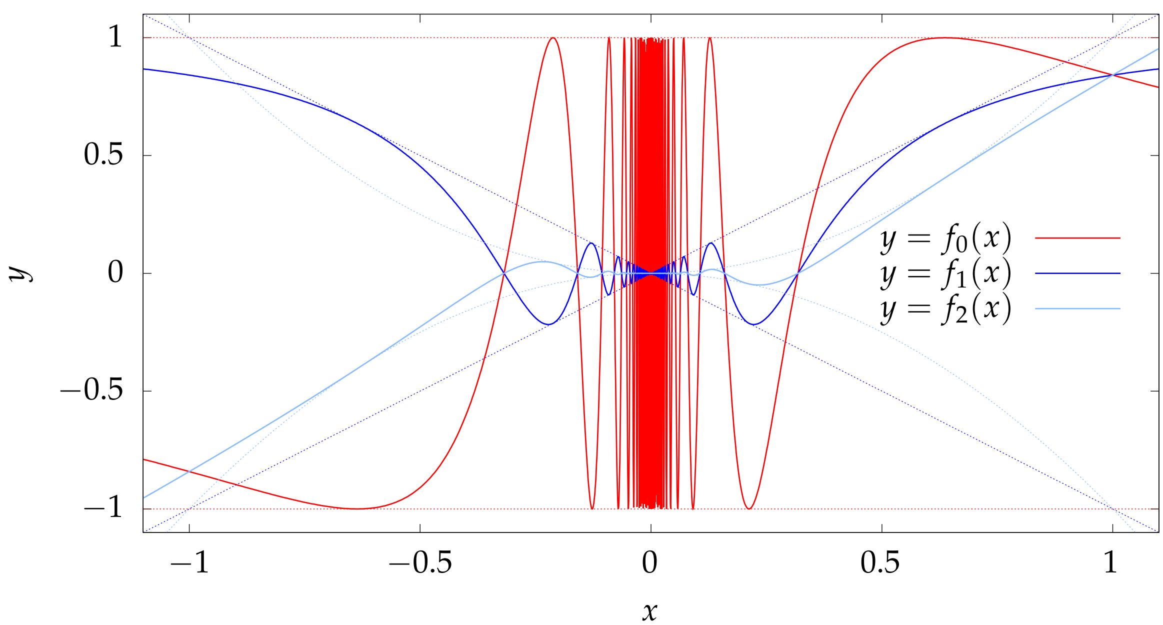

Figure 4: Graph for question 3, showing a sequence of functions with different continuity and differentiability properties at $x = 0$.

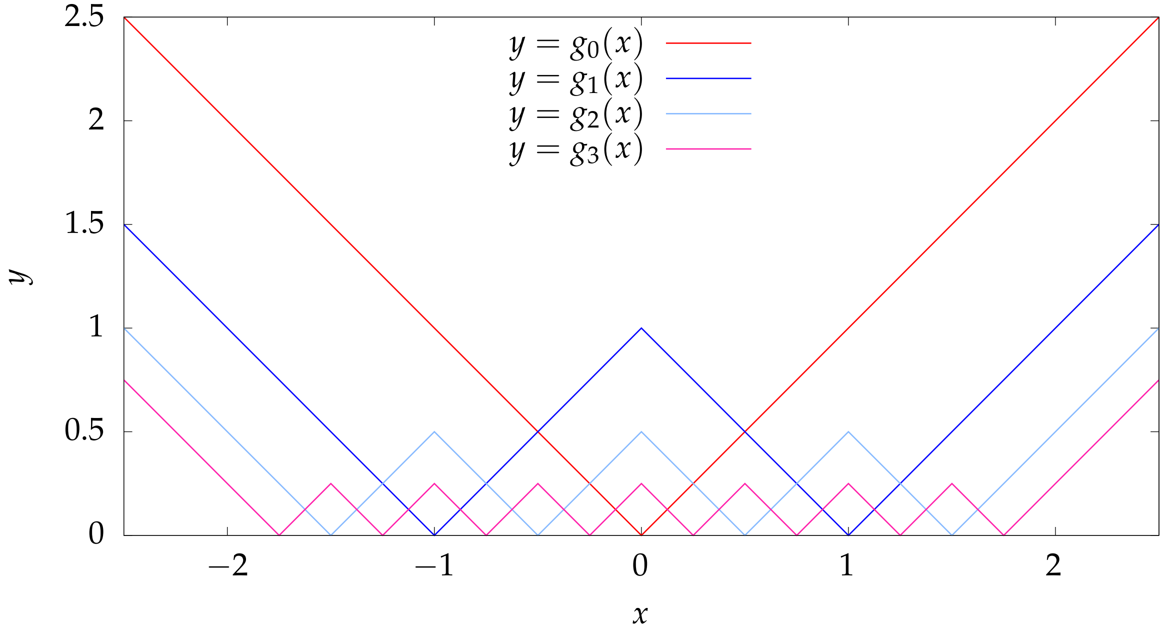

Figure 5: Graphs for question 4, showing a recursively defined sequence of functions.

(b) Since $|\sin(x)| \leq 1$ for all x, the functions f_n can be bounded within the regions $|y| \leq |x|^n$, shown by the dashed lines in Fig. 4. By looking at the graph, it can be seen that over the interval $[-1/2, 1/2]$, these regions become smaller and smaller. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $(1/2)^n < \epsilon$ for $n > N$, so that f_n will lie wholly within the strip $|x - 0| < \epsilon$. Hence f_n will converge uniformly to $f(x) = 0$.

4. (a) Graphs of g_0, g_1, g_2, and g_3 are shown in Fig. 5.

(b) Firstly, note that g_0 is an even function, and since g_n is even if g_{n+1} is even, then by induction g_n is even for all $n \in \mathbb{N}$.

By looking at curves in Fig. 6, it appears that each successive g_n becomes closer to zero. Define the hypothesis P_n to be that $0 \leq g_n(x) \leq 2^{1-n}$ for $x \in [0, 2]$. Since $g_0 = |x|$, it is clear that P_0 is true. Now assume P_n is true and consider P_{n+1}. Then

$$-2^{-n} \leq g_n(x) - 2^{-n} \leq 2^{1-n} - 2^{-n}$$

so

$$-2^{-n} \leq g_n(x) - 2^{-n} \leq 2^{-n}$$

and hence $|g_n(x) - 2^{-n}| \leq 2^{-n}$. Therefore $0 \leq g_{n+1}(x) \leq 2^{-n}$ and P_{n+1} is true. By induction, P_n is true for all $n \in \mathbb{N} \cup \{0\}$.

Now consider when $x > 2$. By looking at Fig. 6, it appears as though the curves are straight lines with slope 1 in this region. Define the hypothesis S_n to be that $g_n(x) = x - 2 + 2^{1-n}$. The case for S_0 is true. Now assume S_n is true and consider S_{n+1}. Then

$$\begin{aligned} g_{n+1}(x) &= |g_n(x) - 2^{-n}| \\ &= |x - 2 + 2^{1-n} - 2^{-n}| \\ &= |x - 2 + 2^{-n}| \\ &= x - 2 + 2^{-n} \end{aligned}$$

where on the final line, the absolute value sign is dropped since it is known the expression will be positive for $x > 2$. Hence S_{n+1} is true. By mathematical induction, S_n is true for all $n \in \mathbb{N} \cup \{0\}$.

Based on these results, it appears that g_n converges uniformly to g defined as

$$g(x) = \begin{cases} |x| - 2 & \text{for } |x| > 2, \\ 0 & \text{for } |x| \leq 2. \end{cases}$$

To show this, consider $\epsilon > 0$. Then since 2^{1-n} converges to 0, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|2^{1-n}| < \epsilon$. For $0 \leq x \leq 2$, and $n \geq \mathbb{N}$

$$|g_n(x) - g(x)| = |g_n(x)| \leq 2^{1-n} < \epsilon.$$

For $x > 2$,

$$|g_n(x) - g(x)| = |x - 2 + 2^{1-n} - (x - 2)| = 2^{1-n} < \epsilon.$$

Since the functions are even, it follows that $|g_n(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$. Hence $g_n \to g$ uniformly.

5. This question involves looking at two special cases of the fourth midterm question.

   (a) Figure 6(a) shows a plot of the several of the f_n for the case of $f(x) = 1 - x$. To prove that these functions do not converge uniformly to f, it must be shown that there exists some $\epsilon > 0$ such that for all N, there exists an integer $n > N$ and $x \in (0, 1)$ such that $|f(x) - f_n(x)| \geq \epsilon$.

   Consider $\epsilon = 1/2$. For any N, choose an integer $n > N$, and then consider $x = 1/2n$. Hence

   $$f(x) - f_n(x) = (1 - x) - 0 = 1 - \frac{1}{2n} > \frac{1}{2},$$

   and thus f_n does not converge to f uniformly.

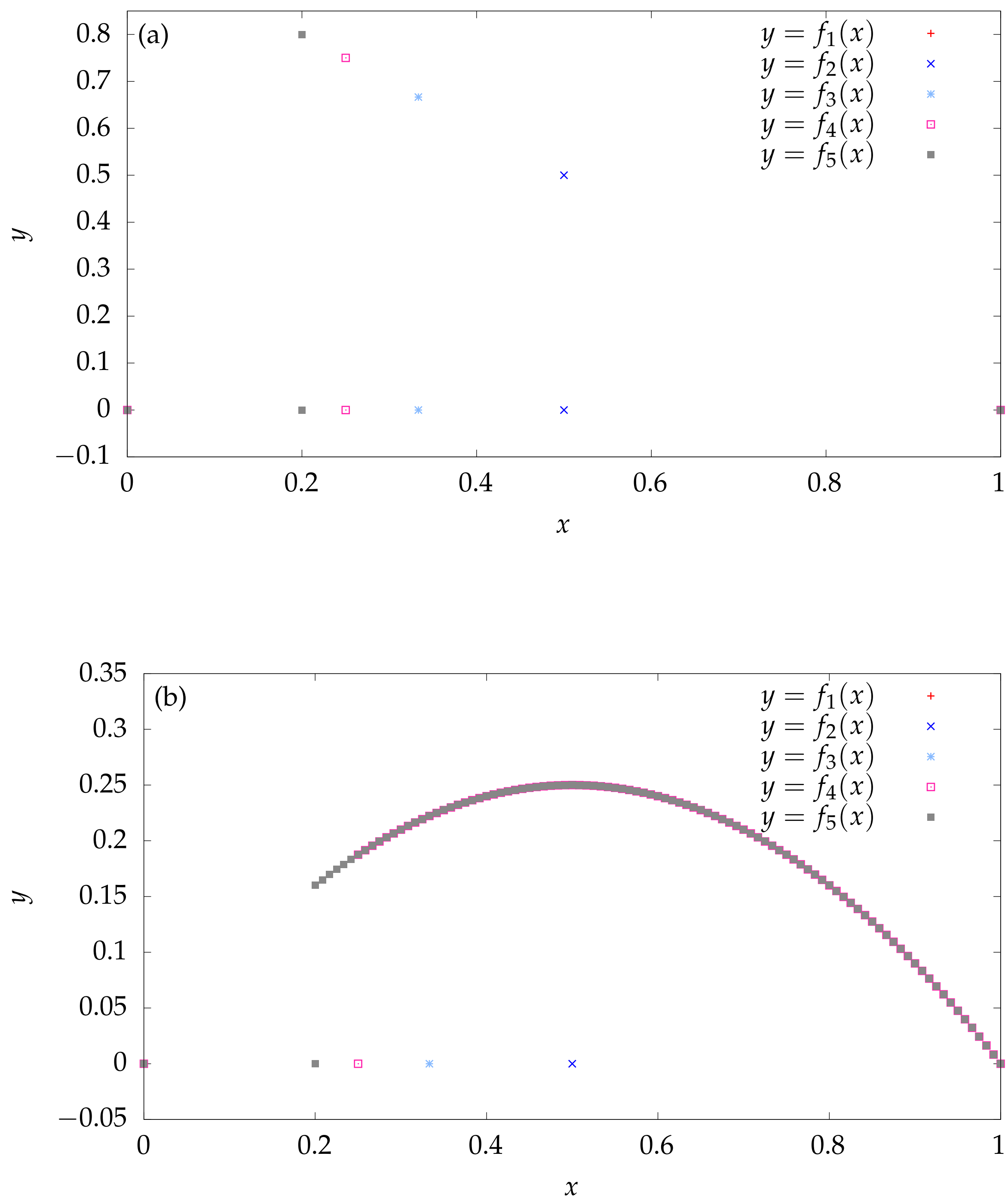

Figure 6: Graphs for question 5 on uniform continuity shown for the case of (a) $f(x) = 1 - x$, and (b) $f(x) = x - x^2$.

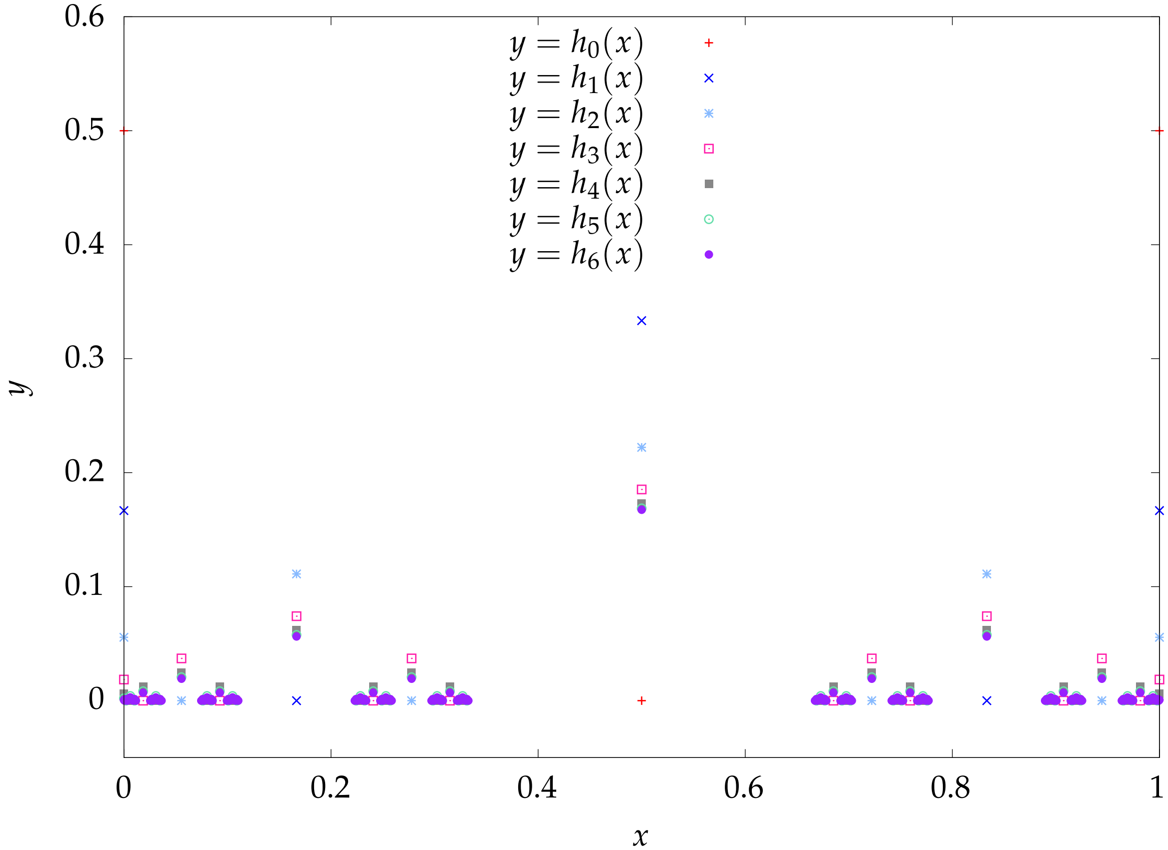

Figure 7: Graphs for question 6, showing a recursively defined sequence of functions.

(b) Figure 6(b) shows a plot of several f_n for the case of $f(x) = x - x^2$. Note that

$$f_n(x) - f(x) = \begin{cases} x - x^2 & \text{if } x < 1/n, \\ 0 & \text{if } x \geq 1/n. \end{cases}$$

For $x \in (0,1)$, the function can be bounded according to $0 < x - x^2 < x$. Hence, by considering the above equation, for any $x \in (0,1)$, $|f_n(x) - f(x)| < 1/n$. Consider any $\epsilon > 0$. There exists an N such that $n > N$ implies $1/n < \epsilon$, and thus $|f_n(x) - f(x)| < \epsilon$ for all $x \in (0,1)$. Hence f_n converges uniformly to f.

6. Despite having a superficial similarity to question 4, the functions in this question have a much more complicated limit. Figure 6 shows plots of several of the h_n, suggesting that the limit is a continuous, fractal curve. The curve appears to be related to the Cantor set, discussed in Example 5 of Ross chapter 13.

To begin, consider proving that the sequence of curves is uniformly Cauchy. Note that all of the h_n are positive, so $|h_n(x)| = h_n(x)$. By applying the triangle inequality

$$h_{n+1}(x) = |h_n(x) - 3^{-(n+1)}| \leq |h_n(x)| + |3^{-(n+1)}| = h_n(x) + 3^{-(n+1)}$$

and thus

$$h_{n+1}(x) - h_n(x) \leq 3^{-(n+1)}. \tag{1}$$

By the reverse triangle inequality (Ross exercise 3.5(b)),

$$\begin{aligned} h_{n+1}(x) &= |h_n(x) - 3^{-(n+1)}| \\ &\geq \Big| |h_n(x)| - |3^{-(n+1)}| \Big| \\ &= |h_n(x) - 3^{-(n+1)}| \\ &\geq h_n(x) - 3^{-(n+1)} \end{aligned}$$

and thus

$$h_{n+1}(x) - h_n(x) \geq -3^{-(n+1)}. \tag{2}$$

Combining Eqs. 1 and 2 gives

$$|h_{n+1}(x) - h_n(x)| \leq 3^{-(n+1)}.$$

Now consider any integer $m > n$. By applying the triangle inequality multiple times, and using the above equation,

$$|h_m(x) - h_n(x)| \leq \sum_{k=n+1}^{m} 3^{-k} = 3^{-(n+1)} \sum_{k=0}^{m-n-1} 3^{-k} < 3^{-(n+1)} \sum_{k=0}^{\infty} 3^{-k} = \frac{3^{-n}}{2}.$$

Thus, since $\lim_{n\to\infty} 3^{-n} = 0$, it follows that for any $\epsilon > 0$, there exists an N such that $m > n \geq N$ implies that $|h_m(x) - h_n(x)| < \epsilon$. Hence the sequence of functions is uniformly Cauchy and thus uniformly convergent. Since each of the h_n is continuous, it follows that the limit h is continuous.

To prove the differentiability properties, several intermediate results are first considered. From the graph, it appears that $h(x)$ vanishes if x is in the Cantor set. To prove this, consider an $x \in [0,1]$, and consider writing a trinary expansion of the form

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where each a_k is either 0, 1, or 2. The existence of such expansions is discussed in Ross chapter 16, although here, due to the properties of the function being considered, base 3 is used instead of the usual base 10. Since

$$\frac{1}{2} = \sum_{k=1}^{\infty} 3^{-k}$$

it follows that

$$h_0(x) = \left| \sum_{k=1}^{\infty} (a_k - 1)3^{-k} \right|.$$

Now suppose that x is in the Cantor set. It can be written in a trinary expansion where the a_k are all either 0 or 2. Note that this expansion is not always unique, since an expansion that ends in an infinite sequence of the form $0222222\ldots$ is equivalent to $1000000\ldots$, and an expansion of the form $1222222\ldots$ is equivalent to $2000000\ldots$[1]. If $a_1 = 2$, then

$$h_0(x) = \frac{1}{3} + \sum_{k=2}^{\infty} (a_k - 1)3^{-k}$$

and if $a_1 = 0$ then

$$h_0(x) = \frac{1}{3} - \sum_{k=2}^{\infty} (a_k - 1)3^{-k},$$

so in general

$$h_0(x) = \frac{1}{3} + (a_1 - 1) \sum_{k=2}^{\infty} (a_k - 1)3^{-k}.$$

Hence

$$\begin{aligned} h_1(x) &= \left| h_0(x) - \frac{1}{3} \right| \\ &= \left| \sum_{k=2}^{\infty} (a_k - 1)3^{-k} \right| \\ &= \frac{1}{3^2} + (a_1 - 1) \sum_{k=3}^{\infty} (a_k - 1)3^{-k}. \end{aligned}$$

Mathematical induction can be applied to show that

$$h_n(x) = \frac{1}{3^{n+1}} + (a_{n+1} - 1) \sum_{k=n+2}^{\infty} (a_k - 1)3^{-k}. \tag{3}$$

Since

$$|h_n(x)| \le \sum_{k=n+1}^{\infty} 3^{-k} = \frac{1}{3^{n+1}} \frac{1}{1 - 1/3}$$

it follows that $h(x) = \lim_{n\to\infty} h_n(x) = 0$.

Now consider a trinary expansion that contains at least one 1, and let the first occurence be at the jth position. If $j = 1$, then $x \in [1/3, 2/3]$, and hence $h_0(x) = |x - 1/2| \in [0, 1/6]$. Thus

$$h_1(x) = \left| h_0(x) - \frac{1}{3} \right| = \frac{1}{3} - h_0(x) \in \left[\frac{1}{6}, \frac{1}{3} \right].$$

[1] This is the same principle by which 1 and $0.9999\ldots$ are decimal expansions of the same real number.

Since $\sum_{k=2}^{\infty} 3^{-k} = 1/6$, the displacements caused by each h_n are not enough to switch the sign of this number. In general for $n \geq 2$

$$h_n(x) = \frac{1}{3} - h_0(x) - \sum_{k=2}^{n} 3^{-k}$$

and hence

$$h(x) = \frac{1}{6} - h_0(x) = \frac{1}{6} - \left| x - \frac{1}{2} \right|.$$

This agrees with the shape of the functions in Fig. 7. Now suppose that the first 1 in the trinary expansion occurs at some position $j > 1$. By following the above argument to obtain Eq. 3,

$$h_{j-2}(x) = \frac{1}{3^{j-1}} + (a_{j-1} - 1) \sum_{k=j}^{\infty} (a_k - 1) 3^{-k}$$

and hence

$$h_{j-1}(x) = \left| \sum_{k=j}^{\infty} (a_k - 1) 3^{-k} \right| = \left| \sum_{k=j+1}^{\infty} (a_k - 1) 3^{-k} \right| \tag{4}$$

which can be rewritten as $h_{j-1}(x) = |y|$ where

$$y = x - \frac{1}{2} - \sum_{k=1}^{j-1} (a_k - 1) 3^{-k}. \tag{5}$$

It can be seen that

$$|y| \in \left[0, \frac{1}{2 \cdot 3^{j+1}} \right]$$

and thus

$$h_j(x) = \frac{1}{3^j} - |y|.$$

Following similar steps as the case for $j = 1$, it can be seen that

$$h(x) = \frac{1}{2 \cdot 3^j} - |y|. \tag{6}$$

With an explicit representation of the function for all values in $[0, 1]$, it is now possible to compute the differentiability properties of h. To begin, suppose that x is in the Cantor set, and write

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}$$

where the a_k are either 0 or 2. Then define the sequence (s_n) according to $s_n = x + 2(1 - a_n) 3^{-n}$ – this sequence converges to x. It can be verified that s_n flips the

nth digit in the trinary expansion from a 0 to 2 or vice versa. This is in the Cantor set so $h(s_n) = 0$, and hence

$$\lim_{n\to\infty} \frac{h(x) - h(s_n)}{x - s_n} = 0.$$

Now consider the sequence t_n defined as

$$t_n = \sum_{k=1}^{n} a_k 3^{-k} + \sum_{k=n+1}^{\infty} 3^{-k}.$$

This corresponds to taking the first n positions to the trinary expansion, followed by an infinite sequence of 1's. By reference to Eq. 4 it can be seen that

$$h_n(t_n) = \left| \sum_{k=j+2}^{\infty} (a_k - 1) 3^{-k} \right| = 0.$$

Hence $h_{n+1}(t_n) = 3^{-(n+1)}$, and $h(t_n) = \frac{1}{2} 3^{-(n+1)}$. Note that

$$|x - t_n| = \sum_{k=n+1}^{\infty} (a_k - 1) 3^{-k} \leq \frac{3^{-n}}{2}$$

and hence

$$\left| \frac{h(x) - h(t_n)}{x - t_n} \right| \geq \frac{1}{3}.$$

If all terms of this form are at least $1/3$ in magnitude, they cannot converge to zero. Hence the limit

$$\lim_{y\to x} \frac{h(x) - h(y)}{x - y}$$

is undefined and h is not differentiable at x. Now suppose that x is not in the Cantor set, with trinary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}.$$

Let j be the smallest value such that $a_j = 1$, and define

$$u = \sum_{k=1}^{j} a_k 3^{-k}, \qquad v = \sum_{k=1}^{j} a_k 3^{-k} + \sum_{k=j+1}^{\infty} 2 \cdot 3^{-k}.$$

These numbers correspond to replacing all digits in the trinary expansions after the jth digit with zeros and twos respectively. Since x is not in the Cantor set, $x \in (u, v)$. By reference to Eqs. 5 and 6, the function is

$$h(x) = \frac{1}{2 \cdot 3^j} - \left| x - \frac{1}{2} - \sum_{k=1}^{j-1} (a_k - 1) 3^{-k} \right|$$

in this interval. Thus $h(x)$ is differentiable apart from when

$$x = \frac{1}{2} + \sum_{k=1}^{j-1} (a_k - 1)3^{-k},$$

which occurs if any only if $a_k = 1$ for all $k > j$.

This completes the proof. h is not differentiable at any point in the Cantor set, nor at any point with a trinary expansion that ends in an infinite sequence of 1's. Otherwise h is differentiable. Interestingly, while the Cantor set is uncountable, the points that have trinary expansions ending in an infinite sequence of 1's are countable.

7. (a) The derivative of $h(x) = \sqrt{x}$ is given by

$$\begin{aligned} h'(x) &= \lim_{y \to x} \frac{h(y) - h(x)}{y - x} \\ &= \lim_{y \to x} \frac{\sqrt{y} - \sqrt{x}}{(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})} \\ &= \lim_{y \to x} \frac{1}{\sqrt{y} + \sqrt{x}} \\ &= \frac{1}{2} x^{-1/2}. \end{aligned}$$

Since the limit exists and is finite for $x > 0$, $h'(x)$ is differentiable for $x > 0$.

(b) The derivative of $f(x) = x^{1/3}$ is given by

$$\begin{aligned} f'(x) &= \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \\ &= \lim_{y \to x} \frac{y^{1/3} - x^{1/3}}{(y^{1/3} - x^{1/3})(y^{2/3} + x^{1/3}y^{1/3} + x^{2/3})} \\ &= \lim_{y \to x} \frac{1}{y^{2/3} + x^{1/3}y^{1/3} + x^{2/3}} \\ &= \frac{1}{3} x^{-2/3}. \end{aligned}$$

Since the limit exists and is finite for $x \neq 0$, $f'(x)$ is differentiable for $x \neq 0$.

(c) f is not differentiable at 0, because even though the limit above exists for $x = 0$, it is $+\infty$. A function is differentiable if and only if the limit exists and is finite.

8. (a) Consider any $\epsilon > 0$, and let $\delta = \sqrt{\epsilon}$. Then for $|x - 0| < \delta$,

$$|f(x) - f(0)| = |f(x)| \leq |x|^2 < \epsilon$$

and thus f is continuous at $x = 0$.

(b) Consider $x \neq 0$. If $x \in \mathbb{Q}$, then consider the sequence $s_n = x + \sqrt{2}/n$, so that $s_n \to x$. Since all s_n are irrational, $f(s_n) = 0$, so $f(s_n) \to 0$. However $f(x) = x^2 \neq 0$, so f is not continuous at x.

If $x \notin \mathbb{Q}$, then define a sequence s_n so that $s_n \in (x - n^{-1}, x + n^{-1})$ and $s_n \in \mathbb{Q}$. By the "Denseness of $\mathbb{Q}$", such a choice is always possible. Observe that $s_n \to x$. However $f(s_n) = s_n^2$, so $f(s_n) \to x^2$, but $f(x) = 0 \neq x^2$. Hence f is not continuous at x.

(c) Since

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq \left| \frac{x^2}{x} \right| \leq |x|$$

then it follows that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

and thus f is differentiable at $x = 0$.