Math 104: Homework 7 solutions

1. (a) The derivative of $f(x) = \sqrt{x}$ is

$$f'(x) = \frac{1}{2\sqrt{x}}$$

which is unbounded as $x \to 0$. Since $f(x)$ is continuous on $[0,1]$, it is uniformly continous on this interval by Theorem 19.2. Hence for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [0,1]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since this property is still satisfied if x and y are chosen from $(0,1]$, then $f(x)$ is uniformly continuous on $(0,1]$ also.

(b) Choose a $\epsilon > 0$, and consider $x, y \in [1, \infty)$. Then

$$\begin{aligned} |f(x) - f(y)| &= |\sqrt{x} - \sqrt{y}| \\ &= \left| (\sqrt{x} - \sqrt{y}) \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \right| \\ &= \frac{|x - y|}{|\sqrt{x} + \sqrt{y}|} \\ &\leq \frac{|x - y|}{2} \end{aligned}$$

where the final line makes use of the inequality $\sqrt{x} \geq 1$ for $x \geq 1$. Hence for $|x - y| < \delta$ where $\delta = 2\epsilon$, then

$$|f(x) - f(y)| < \epsilon$$

and thus f is uniformly continuous on $[1, \infty)$.

2. (a) Choose $\epsilon > 0$. Then since g is uniformly continuous, there exists a $\delta > 0$, such that for all $a, b \in \mathbb{R}$ with $|a - b| < \delta$,

$$|g(a) - g(b)| < \epsilon.$$

Similarly, since f is uniformly continuous, there exists κ such that for all $x, y \in S$ with $|x - y| < \kappa$,

$$|f(x) - f(y)| < \delta.$$

Hence, by equating $a = f(x)$ and $b = g(y)$, it can be seen that for all $|x - y| < \kappa$, where $x, y \in S$,

$$|g(f(x)) - g(f(y))| < \epsilon$$

and thus $g \circ f$ is uniformly continuous.

(b) Choose $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for all $x, y \in S$ where $|x-y| < \delta_1$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2}$$

and there exists $\delta_2 > 0$ such that for all $x, y \in S$ where $|x-y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\epsilon}{2}.$$

Now consider any $x, y \in S$ satisfying $|x-y| < \delta$, where $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| &= |(f(x) - f(y)) - (g(x) - g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $f+g$ is uniformly continuous on S.

(c) Consider $f(x) = x$ on $\mathbb{R}$. Then for any $\epsilon > 0$, it can be seen that for any $x, y \in \mathbb{R}$ satisfying $|x-y| < \delta$ where $\delta = \epsilon$, then $|f(x) - f(y)| < \epsilon$. Hence f is uniformly continuous on $\mathbb{R}$.

Now consider $f(x) = x$ and $g(x) = x$. Then the multiplication is $h(x) = f(x) \cdot g(x) = x^2$. To show that h is not continuous, pick $\epsilon = 1$, and consider any $\delta > 0$. If $x = \delta^{-1} + \frac{\delta}{2}$ and $y = \delta^{-1}$, then $|x-y| = \frac{\delta}{2}$ but

$$|h(x) - h(y)| = \left| \left(\frac{1}{\delta} + \frac{\delta}{2} \right)^2 - \frac{1}{\delta^2} \right| = 1 + \frac{\delta^2}{4} > 1$$

Hence there does not exist a $\delta > 0$ such that for all $|x-y| < \delta$, $|h(x) - h(y)| < 1$, so h is not continous on $\mathbb{R}$.

3. (a) Figure 1 shows a graph of the function $f(x) = (x-1)^{-1}(x-2)^{-2}$.

(b) Consider a sequence (a_n) with terms in $(2,3)$ which converges to 2. Then any term can be written at $2 + \lambda$ for some $\lambda \in (0,1)$, and

$$f(2+\lambda) = \frac{1}{((1+\lambda)\lambda^2} > \frac{1}{2\lambda^2}$$

Consider any $M > 0$. Then there exists an $N \in \mathbb{N}$ such that $a_n < 2 + 1/\sqrt{2M}$ for all $n > N$. Then $f(a_n) > M$ for all $n > N$, and thus $\lim_{x \to 2^+} f(x) = \infty$. Similar arguments show that

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \infty \\ \lim_{x \to 1^+} f(x) &= \infty \\ \lim_{x \to 1^-} f(x) &= -\infty \end{aligned}$$

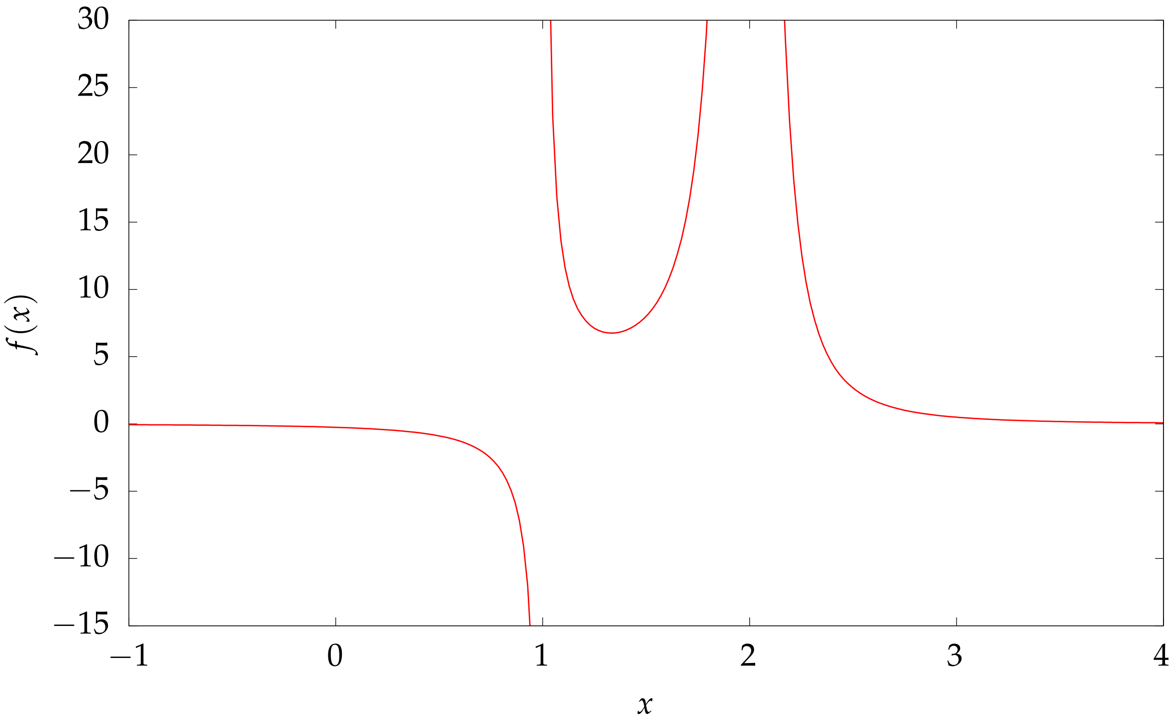

Figure 1: A graph of the function $f(x) = (x-1)^{-1}(x-2)^{-2}$.

(c) By Theorem 20.10, a limit at a point is well defined if and only if the positive and negative limits are equal. Hence $\lim_{x \to 2} f(x) = \infty$, and $\lim_{x \to 1} f(x)$ is undefined.

4. (a) Suppose that $f_1(x) \leq f_2(x)$ for all $x \in (a, b)$, but $L_1 > L_2$. Then $L_1 = L_2 + \Delta$ for some $\Delta > 0$. Now consider the sequence $c_n = a + (b-a)/(2n)$ which converges to a. Since $\lim_{x \to a^+} f_1(x) = L_1$, there exists an N_1 such that $n > N_1$ implies

$$|f_1(c_n) - L_1| < \frac{\Delta}{2}$$

and hence $f_1(c_n) - L_1 > -\Delta/2$, so that $f_1(c_n) > L_1 - \Delta/2$. Similarly there exists an N_2 such that $n > N_2$ implies

$$|f_2(c_n) - L_2| < \frac{\Delta}{2}$$

and hence $f_2(c_n) - L_2 < \Delta/2$, so that $f_2(c_n) < L_2 + \Delta/2$. Bu since $\Delta = L_1 - L_2$, then $L_1 - \Delta/2 = L_2 + \Delta/2$, and hence $f_1(c_n) < f_2(c_n)$ which is a contradiction. Hence $L_1 \leq L_2$.

(b) Consider $f_1(x) = 0$, and $f_2(x) = x$. Then for all $x \in (0,1)$, $f_1(x) < f_2(x)$. However $\lim_{x\to 0^+} f_1(x) = 0$ and $\lim_{x\to 0^+} f_2(x) = 0$, so $L_1 = L_2$.

5. (a) At $x = 1$, the series becomes $\sum a_n$. In order for this series to converge, then $\lim a_n = 0$. However, if the sequence (a_n) has infinitely many non-zero integers, then there does not exist an N such that $n > N$ implies $|a_n - 0| < 1/2$. Hence the series does not converge at $x = 1$, so the radius of convergence must be less than or equal to 1.

(b) Suppose that $\limsup |a_n| = a > 0$. Then there exist infinitely many terms a_{n_k} such that $|a_{n_k}| > a/2$. Now consider the sequence with terms $|a_n|^{1/n}$. This has a subsequence $|a_{n_k}|$, which satisfies

$$|a_{n_k}|^{1/n_k} > \left(\frac{a}{2}\right)^{1/n_k}$$

and as $n_k \to \infty$, $|a_{n_k}|^{1/n_k} \to 1$. Hence, $\limsup |a_n|^{1/n} \geq 1$.

6. (a) For a fixed value of $x \in [0, \infty)$,

$$\lim_{n\to\infty} \frac{x}{n} = x \cdot \lim_{n\to\infty} \frac{1}{n} = 0.$$

and hence the sequence of functions converges pointwise to $f(x) = 0$.

(b) Choose $\epsilon > 0$. Then let $N = \epsilon^{-1}$. If $n > N$, then for all $x \in [0,1]$,

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\right| \leq \frac{1}{n} < \epsilon.$$

and hence $f_n \to f$ uniformly on $[0,1]$.

(c) Pick $\epsilon = 1$. To prove that f_n does not tend to f uniformly on $[0,\infty)$, it must be shown that there does not exist an N such that $n > N$ implies $|f_n(x) - f(x)| < 1$ for all $x \in [0,\infty)$. However, for any n, if $x = n$, then

$$|f_n(x) - f(x)| = \left|\frac{n}{n} - 0\right| = 1.$$

Hence for all n there exists an x such that $|f_n(x) - f(x)| \geq 1$, so f_n does not converge uniformly to f.

7. (a) For $x = 0$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{0}{1} = 0.$$

For a fixed value of $x \in (0,\infty)$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1 + nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x^2} = \frac{x}{x^2} = \frac{1}{x}.$$

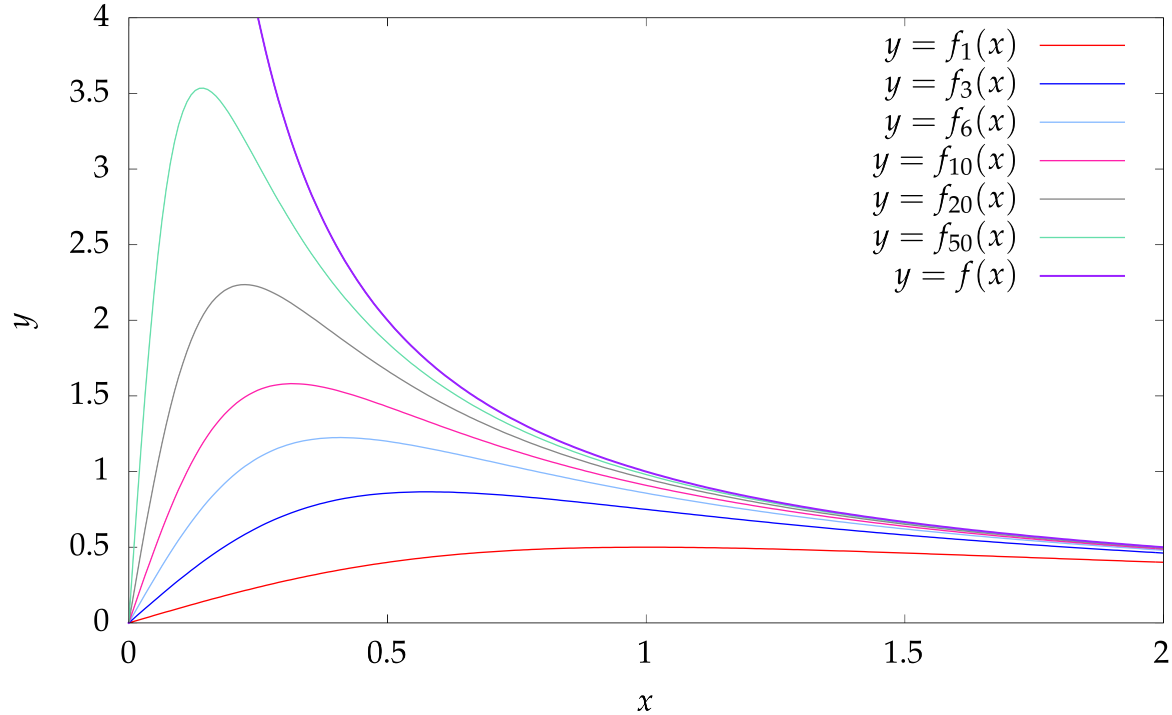

Figure 2: Graphs of the function $f_n(x) = nx/(1 + nx^2)$ for several values of n, as well as its pointwise limit $f(x) = x^{-2}$.

Hence on the interval $[0, \infty)$, f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{if } x > 0. \end{cases}$$

(b) To show that f_n does not converge to f uniformly on $[0, 1]$, consider $x = \frac{1}{n}$ for f_n:

$$|f_n(x) - f(x)| = \left| \frac{1}{1 + \frac{1}{n}} - n \right| = \left| n - \frac{n}{n+1} \right|$$

The fraction $n/(n + 1)$ is smaller than 1 for all $n \in \mathbb{N}$. Hence for $n \geq 2$, and $x = 1/n$,

$$|f_n(x) - f(x)| > 1.$$

Hence there does not exist an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < 1$ for all $x \in [0, 1]$.

(c) For $x \in [1, \infty)$,

$$|f_n(x) - f(x)| = \left| \frac{nx}{1 + nx^2} - \frac{1}{x} \right| = \left| \frac{1}{(1 + nx^2)x} \right| < \frac{1}{n}.$$

Thus for any $\epsilon > 0$, if $N = \epsilon^{-1}$, then $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [1, \infty)$, and hence $f_n \to f$ uniformly on this interval.

8. Suppose that $f(I)$ is open for any open interval I, but that f is not monotonic. Then there would exist some interval $[a, b]$ over which it is non-monotonic.

   Suppose $f(a) = f(b)$. Then if it is non-monotonic, it is non-constant so there exists an $x \in (a, b)$ such that $f(x) \neq f(a)$. Consider $f([a, b])$: by Corollary 18.3, the set must be an interval, and by Theorem 18.1 it must be bounded and attain its bounds, thus being some closed interval $[c, d]$. If $f(x) > f(a)$, then $d > f(a)$, in which case $f((a, b))$ must still contain d, and thus this set is not open since d is non an interior point. If $f(x) < f(b)$, then $c < f(a)$, in which case $f((a, b)$ must still contain c, and thus this set is not open. Either possibility leads to a contradiction.

   Now suppose that $f(a) < f(b)$. Then, by the argument above, $f([a, b])$ must be a closed interval $[c, d]$. If $d > f(b)$, then d is still in $f((a, b))$, and thus this set is not open. Hence, since $d \geq f(b)$, then the only remaining possibility is $d = f(b)$, and hence $f(x) \leq f(b)$ for all $x \in [a, b]$. Similarly, it can be shown that $c = f(a)$, and hence $f(a) \leq f(x)$ for all $x \in [a, b]$.

   Now consider the interval $[a, x]$. If $f(a) = f(x)$, this leads to a contradiction following the same argument above. Hence $f(a) < f(x)$, and thus, by applying the argument above, $f(y) \leq f(x)$ for all $y \in [a, b]$.

   Since x and y are chosen arbitrarily, it has been shown that for all $x, y \in [a, b]$ where $x \leq y$, then $f(x) \leq f(y)$, showing that the function non-decreasing, and hence monotonic. If $f(a) > f(b)$, the above arguments can be applied to $-f$, to show that f is non-increasing, and hence monotonic also. Either possibility violates the original assumption the f is non-monotonic.

   All possibilities lead to a contradiction, so the original assumption must be false. Hence if $f(I)$ is open for any open interval I, then f is monotonic.