Math 104: Homework 6 solutions

1. To find the interior of A, consider any element, which can be written as $1/n$ for some $n \in \mathbb{N}$. For any $\epsilon > 0$, there exists an $m \in \mathbb{N}$ such that $1/m < \epsilon$. Consider the point $x = 1/n + 1/(\sqrt{2}m)$. Then $x \in N_\epsilon(1/n)$, but $x \notin \mathbb{Q}$ so $x \notin A$. Hence no neighborhood of $1/n$ is contained in A. Hence the interior of A is $\emptyset$.

   Now consider the interior of B. If $x \in (0,1)$, then $N_r(x) \subseteq B$ if $r = \min\{x, 1-x\}$. Consider any element $x \in B$ where $x \geq 1$. Then x is rational. Choose any $\epsilon > 0$, and find an $n \in \mathbb{N}$ such that $1/n < \epsilon$. But since $x + 1/(\sqrt{2}n)$ is irrational and not an element of $N_\epsilon(1)$, then x is not in the interior of B. A similar argument can be used to show that all $x \in B$ satisfying $x \leq 0$ are not interior points. Hence the interior of B is $(0,1)$.

   Now consider the closure of A. Since each element of A is isolated, all elements are isolated, and are not limit points. For any value $x \neq 0$, and $x \notin A$, then x is a finite distance from any element of A and hence x is not a limit point. Finally, consider $x = 0$. For any neighborhood $N_\epsilon(0)$, there exists an $n \in N$ such that $1/n < \epsilon$ and hence $1/n \in N_\epsilon(0)$. Hence 0 is a limit point. The closure of A is therefore $A \cup \{0\}$.

   Finally, consider the closure of B. Consider any real number r such that $r \notin B$. Since any neighborhood $N_\epsilon(r)$ contains a rational number, then $N_\epsilon(r) \cap B$ is non-empty and hence r must be a limit point of B. Thus the closure of B is $\mathbb{R}$.

2. First, consider the function at $x = 0$. Choose $\epsilon > 0$. Then

   $$|h(x) - h(0)| = |h(x) - 0| < |x|.$$

   Hence, if $\delta = 0$, then $|x - 0| < \delta$ implies $|h(x) - h(0)| < \epsilon$. Hence h is continuous at 0.

   Now consider $x \neq 0$. Suppose x is irrational. Then define a sequence (a_n) where a_n is a rational number in the range $(x - 1/n, x + 1/n)$; by Theorem 4.7, "The Denseness of $\mathbb{Q}$", this is always possible. Then $a_n \to x$, and since $h(a_n) = a_n$ for all n, then $h(a_n) \to x$. However, $h(x) = 0$, so h is not continuous at x.

   Now suppose that x is rational. Then define a sequence $a_n = x + \sqrt{2}/n$; all of these terms are irrational, since otherwise it would imply that $\sqrt{2}$ was rational. Then $h(a_n) = 0$ for all n, and hence $h(a_n) \to 0$. However $h(x) = x \neq 0$, so h is not continuous at x.

   Therefore h is continuous only at $x = 0$.

3. (a) Consider $f(x) = x^2$ at $x = 2$. Choose $\epsilon > 0$. Then

   $$|f(x) - f(2)| = |x^2 - 4| = |x - 2| \cdot |x + 2|.$$

For all values of ϵ, the value of δ can be chosen to be less than 1, in which case $|x-2|<1$, so $|x+2|<5$. Hence if $\delta=\min\{1,\epsilon/5\}$, then $|x-2|<\delta$ implies that

$$|f(x)-f(2)|<\frac{\epsilon}{5}\cdot 5=\epsilon$$

so f is continuous at 2.

(b) Consider $f(x)=\sqrt{x}$ at 0. This function has a natural domain $[0,\infty)$, so $x\geq 0$. Then

$$|f(x)-f(0)|=|\sqrt{x}-0|=\sqrt{x}.$$

Hence for any $\epsilon>0$, if $\delta=\epsilon^2$, then $|x-0|<\delta$ implies

$$|f(x)-f(0)|<\sqrt{\epsilon^2}=\epsilon.$$

Hence f is continuous at 0.

(c) Consider $f(x)=x\sin(1/x)$ for $x\neq 0$ and $f(0)=0$. Then

$$|f(x)-f(0)|\leq|x-0|=|x|.$$

Hence, for any $\epsilon>0$, if $\delta=\epsilon$, then $|x-0|<\delta$ implies

$$|f(x)-f(0)|\leq\delta=\epsilon$$

so f is continuous at 0.

(d) Consider $f(x)=x^3$. Then for an arbitrary x_0,

$$|f(x)-f(x_0)|=|x^3-x_0^3|=|x-x_0|\cdot|x^2+xx_0+x_0^2|.$$

By restricting δ to be less than 1, in which case $|x-x_0|<1$, it can be seen that

$$\begin{aligned}|x^2+xx_0+x_0^2| &\leq |x^2|+|x|\cdot|x_0|+|x_0^2|\\ &\leq (|x_0|+1)^2+(|x_0|+1)|x_0|+x_0^2=3x_0^2+3|x_0|+1.\end{aligned}$$

Hence if $K=3x_0^2+3|x_0|+1>0$, then

$$|f(x)-f(x_0)|\leq K|x-x_0|.$$

For any $\epsilon>0$, if $\delta=\epsilon/K$, then $|x-x_0|<\delta$ implies

$$|f(x)-f(x_0)|<K\delta=K\frac{\epsilon}{K}=\epsilon.$$

Hence f is continuous at x_0. Since x_0 is arbitrary, f is continuous on $\mathbb{R}$.

4. Consider the map $f(x)=x^2$. If $x\in(0,1)$, then $f(x)\in(0,1)$. Suppose that the map has a fixed point, so that $f(x)=x$. Then $x^2=x$, so $x(x-1)=0$, and therefore $x=0,1$ are the only solutions. Since these both lie outside $(0,1)$, there are no fixed points.

5. This is false. Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 1. \end{cases}$$

Consider any sequence (a_n) that converges to zero. Since f is even, $f(0 + a_n) - f(0 - a_n) = 0$, so $\lim f(0 + a_n) - f(0 - a_n) = 0$. However, f is not continuous at $x = 0$. To verify this, consider the sequence $a_n = 1/n$. Then $f(a_n) = 0$ for all n and hence $\lim f(a_n) = 0$. However, $\lim a_n = 0$, and $f(0) = 1$.

6. A polynomial of odd degree can be written as

$$p(x) = \sum_{k=0}^{K} a_k x^k$$

where K is odd and $a_K \neq 0$. Assume $a_K > 0$. Then the polynomial can be written as

$$p(x) = x^K \left(a_K + \sum_{k=0}^{K-1} \frac{a_k}{x^{K-k}} \right).$$

Now, consider the sequence

$$s_n = a_K + \sum_{k=0}^{K-1} \frac{a_k}{n^{K-k}}.$$

Since all the terms of the form $1/n^m$ for $m \in \mathbb{N}$ converge to zero, then $\lim s_n = a_K$. Hence, there exists N_1 such that $n > N_1$ implies

$$|s_n - a_K| < a_K$$

and hence $s_n > 0$. Thus

$$p(N_1 + 1) = (N_1 + 1)^K s_{N_1+1} > 0.$$

Similarly, the sequence

$$t_n = a_K + \sum_{k=0}^{K-1} \frac{a_k}{(-n)^{K-k}}$$

converges to a_K, so there exists an N_2 such that $n > N_2$ implies $t_n > 0$. Hence

$$p(-(N_2 + 1)) = -(N_2 + 1)^K s_{N_2+1} < 0.$$

Applying the Intermediate Value Theorem shows that there exists an $x \in (-(N_2 + 1), N_1 + 1)$ such that $p(x) = 0$. Hence the polynomial has a real root. For the case when $a_K < 0$, the same argument can be applied to the polynomial $-p(x)$.

7. Consider $g(x) = f(x+1) - f(x)$ on the interval $[0,1]$. Then $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1) = -g(0)$. If $g(0) = 0$, then setting $x = 0$ and $y = 1$ satisfies $|x - y| = 1$ and $f(x) = f(y)$.

   Otherwise, if $g(0) \neq 0$, then the Intermediate Value Theorem can be applied to the interval $[0,1]$, to show that there exists an $x \in (0,1)$ such that $g(x) = 0$. Putting $y = x + 1$ satisfies $|x - y| = 1$ and $f(x) = f(y)$.

8. Pick a point $x_0 \in (a,b)$. Then there exists a Δ such that $N = (x_0 - 2\Delta, x_0 + 2\Delta) \subseteq (a,b)$. Now define $p = f(x_0 - \Delta)$ and $q = f(x_0 + \Delta)$, and consider $x \in [x_0 - \Delta, x_0]$. By using the convexity property applied to the point x between $x_0 - \Delta$ and x_0,

$$f(x) \geq \frac{f(x_0)(x - x_0 + \Delta) + (x_0 - x)p}{\Delta} = f(x_0) + \frac{(x - x_0)(f(x_0) - p)}{\Delta}$$

   and hence

$$f(x) - f(x_0) \geq \frac{(x - x_0)(f(x_0) - p)}{\Delta}. \tag{1}$$

   Applying the convexity property to the point x_0 between x and $x_0 + \Delta$ shows that

$$f(x_0) \geq \frac{f(x)((x_0 + \Delta) - x_0) + (x_0 - x)q}{x_0 + \Delta - x}$$

   and hence

$$f(x_0)(x_0 + \Delta - x) \geq f(x)\Delta + (x_0 - x)q$$

   so

$$f(x) - f(x_0) \leq \frac{(x - x_0)(q - f(x_0)}{\Delta}. \tag{2}$$

   If $K = \max\{(f(x_0) - p)/\Delta, (q - f(x_0))/\Delta\}$, then combining Eqs. 1 & 2 shows that

$$|f(x) - f(x_0)| \leq K|x - x_0|.$$

   By symmetry, the same arguments can be applied to show that this inequality also holds for $x \in [x_0, x_0 + \Delta]$. From here, it can be seen that for any $\epsilon > 0$, if $\delta = \epsilon/K$, then $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| < K\delta = K\frac{\epsilon}{K} = \epsilon.$$

   Hence f must be continuous at any interior point. To prove that f is continuous at a given x_0, the above proof requires that function values are available on both sides of x_0, which will be not be true at the end points. An example of a convex function that is not continuous at the end points is

$$f(x) = \begin{cases} 1 & \text{if } x \neq a \text{ and } x \neq b, \\ 0 & \text{if } x = a \text{ or } x = b. \end{cases}$$