Math 104: Homework 5 solutions

1. (a) Suppose $p > 1$. Then

$$\begin{aligned}
\sum_{n=2}^{N} \frac{1}{n(\log n)^p} &\leq \frac{1}{2(\log 2)^p} + \int_2^N \frac{1}{x(\log x)^p} dx \\
&\leq \frac{1}{2(\log 2)^p} + \int_{\log 2}^{\log N} \frac{1}{y^p} dy \\
&\leq \frac{1}{2(\log 2)^p} + \left[\frac{1-p}{y^{p-1}} \right]_{\log 2}^{\log N} \\
&\leq \frac{1}{2(\log 2)^p} + (1-p) \left[(\log N)^{1-p} - (\log 2)^{1-p} \right]
\end{aligned}$$

where the substitution $x = \log y$ has been used. Since $1 - p < 0$, $(\log N)^{1-p} \to 0$ as $N \to \infty$. Hence the sum is bounded above, and since all the terms are positive, it must converge. Now suppose $p = 1$. Then

$$\begin{aligned}
\sum_{n=2}^{N} \frac{1}{n(\log n)} &\geq \int_2^{N+1} \frac{1}{x(\log x)} dx \\
&\geq \int_{\log 2}^{\log(N+1)} \frac{1}{y} dy \\
&\geq [\log y]_{\log 2}^{\log(N+1)} \\
&\geq \log \log(N+1) - \log \log 2
\end{aligned}$$

and since $\log \log(N+1)$ is unbounded as $N \to \infty$, the sum must diverge. Now consider the case when $p < 1$. Observe that for $N \geq 3$,

$$\begin{aligned}
\sum_{n=2}^{N} \frac{1}{n(\log n)^p} &= \frac{1}{2(\log 2)^p} + \sum_{n=2}^{N} \frac{1}{n(\log n)^p} \\
&\geq \frac{1}{2(\log 2)^p} + \sum_{n=2}^{N} \frac{1}{n(\log n)}
\end{aligned}$$

which is true since $\log n > 1$ for $n \geq 3$. Since it was previously shown that the sum diverges for $p = 1$, it must diverge for $p < 1$ also.

(b) Choose an $\epsilon > 0$. The Cauchy criterion states that there exists an N such that $n \geq m > N$ implies that

$$\left| \sum_{k=m}^{n} a_n \right| < \frac{\epsilon}{2}.$$

For any $m > N$, set $n = 2m$. Then, by making use of the fact that all the terms are positive, and that if $j > i$ then $a_j \leq a_i$,

$$\frac{\epsilon}{2} > \left| \sum_{k=m}^{2m} a_k \right| \geq \left| \sum_{k=m}^{2m} a_{2m} \right| = (m+1)a_{2m}.$$

Now consider an integer $l > 2N + 1$. If l is even, then $l = 2m$ for some $m > N$, so

$$|la_l| = la_l = 2ma_{2m} < 2(m+1)a_{2m} < \frac{2\epsilon}{2} = \epsilon.$$

If l is odd, then $l = 2m + 1$ for some $m > N$, so

$$|la_l| = la_l = (2m+1)a_{2m+1} < 2(m+1)a_{2m+1} \leq 2(m+1)a_{2m} < \frac{2\epsilon}{2} = \epsilon.$$

Hence $|la_l| < \epsilon$ for all $l > 2N + 1$, and thus $\lim_{n\to\infty} na_n = 0$.

The converse result is not true. Let (a_n) be defined by $a_1 = 1$ and $a_n = 1/(n \log n)$ for $n \geq 2$. Then

$$na_n = \frac{n}{n \log n} = \frac{1}{\log n}$$

which converges to 0 as $n \to 0$. However, as shown in part (a), $\sum 1/(n \log n)$ diverges.

2. • $d_1(x,y) = (x-y)^2$ is not a metric. Suppose $x = 0$, $y = 1$, and $z = 2$. Then

$$d_1(x,z) = 2^2 = 4$$

but

$$d_1(x,y) + d_2(y,z) = 1^1 + 1^2 = 2$$

and thus the triangle inequality is not satisfied.

• $d_2(x,y) = \sqrt{|x-y|}$ is a metric. Consider the three properties of a metric:

M1. If $x \neq y$, then $|x-y| > 0$ and hence $d(x,y) > 0$. If $x = y$, then $d(x,y) = 0$.

M2. $d(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d(y,x)$ and thus it is symmetric.

M3. Suppose that the triangle inequality did not hold. The there exists x, y, and z such that

$$d(x,y) + d(y,z) < d(x,z).$$

Then since both sides are positive, this would imply that

$$(d(x,y) + d(y,z))^2 < d(x,z)^2$$

so that

$$|x-y| + |y-z| + 2\sqrt{|x-y|}\sqrt{|y-z|} < |x-z|.$$

But by making use of the usual triangle inequality, $|x-y|+|y-z| \geq |x-z|$ and hence

$$|x-z| + 2\sqrt{|x-y|}\sqrt{|y-z|} < |x-z|$$

But since $2\sqrt{|x-y|}\sqrt{|y-z|} \geq 0$, this is a contradiction. Hence $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in \mathbb{R}$.

- $d_3(x,y) = |x^2 - y^2|$ is not a metric. $d(1,-1) = |1^1 - (-1)^2| = 0$, but a zero distance is not permissible for distinct elements.
- $d_4(x,y) = |x - 2y|$ is not a metric, since it does not satisfy $d(x,y) = d(y,x)$.

3. (a) Consider the three properties of a metric:

M1. Since d is defined as a supremum of non-negative terms, then $d(\mathbf{x},\mathbf{y}) \geq 0$ for all $\mathbf{x}$ and $\mathbf{y}$. If $d(\mathbf{x},\mathbf{y}) = 0$, then $\sup\{|x_i - y_i| : i \in \mathbb{N}\} = 0$, so $|x_i - y_i| = 0$ for all $n \in \mathbb{N}$.

M2. For all $\mathbf{x}$ and $\mathbf{y}$

$$\begin{aligned} d(\mathbf{x},\mathbf{y}) &= \sup\{|x_i - y_i| : i \in \mathbb{N}\} \\ &= \sup\{|y_i - x_i| : i \in \mathbb{N}\} \\ &= d(\mathbf{y},\mathbf{x}) \end{aligned}$$

and thus d is symmetric.

M3. Consider three sequences $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Then

$$d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) = \sup\{|x_i - y_i| : i \in \mathbb{N}\} + \sup\{|y_i - z_i| : i \in \mathbb{N}\}.$$

Now consider the set

$$\{|x_i - y_i| + |y_i - z_i| : i \in \mathbb{N}\}.$$

Since $\sup\{|x_i - y_i| : i \in \mathbb{N}\}$ is an upper bound for all the $|x_i - y_i|$, and $\sup\{|y_i - z_i| : i \in \mathbb{N}\}$ is an upper bound for all the $|y_i - z_i|$, then the sum of these two bounds must be an upper bound for all terms of the form $|x_i - y_i| + |y_i - z_i|$, and hence

$$\begin{aligned} d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) &\geq \sup\{|x_i - y_i| + |y_i - z_i| : i \in \mathbb{N}\} \\ &\geq \sup\{|x_i - z_i| : i \in \mathbb{N}\} = d(\mathbf{x},\mathbf{z}), \end{aligned}$$

where the usual triangle inequality for real numbers has been used.

(b) d^* does not define a metric since it is not well-defined. Consider $x_n = 1$ and $y_n = 2$ for all n. Then

$$\sum_{n=1}^{N} |x_n - y_n| = N$$

and hence $\sum |x_n - y_n|$ diverges, meaning that d^* does not have a well-defined value.

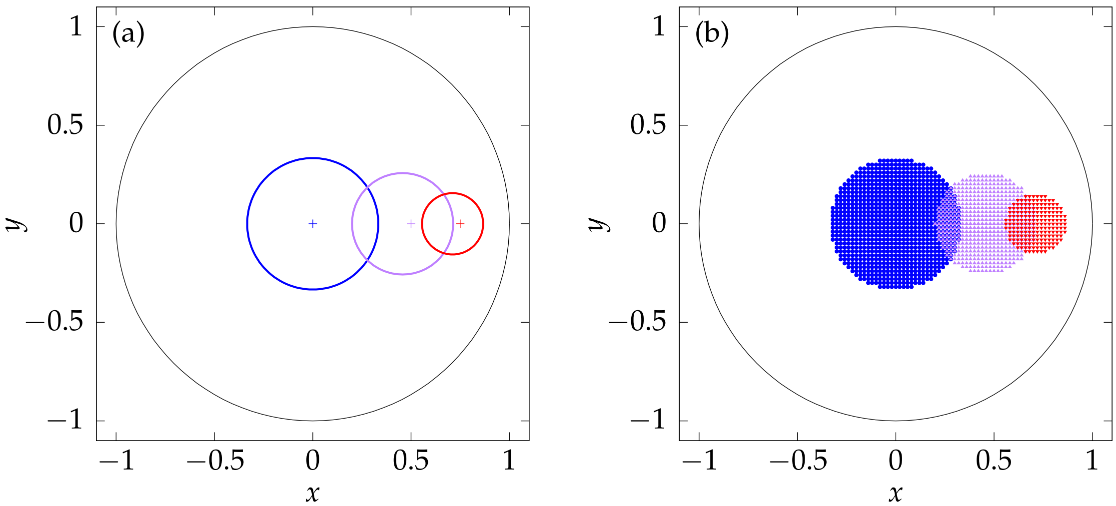

Figure 1: There neighborhoods of radius $\cosh^{-1} 5/4$ in the Poincaré disk, computed (a) analytically, and (b) with a C++ program. The neighborhoods at $(0,0)$, $(1/2,0)$, and $(3/4,0)$ are shown in blue, purple, and red respectively.

4. Write $\mathbf{u} = (\lambda, 0)$ and $\mathbf{v} = (v_1, v_2)$. The neighborhoods are defined by all $\mathbf{v}$ such that $d(\mathbf{u}, \mathbf{v}) < r$, which corresponds to

$$1 + \frac{2||\mathbf{u} - \mathbf{v}||^2}{(1 - ||\mathbf{u}||^2)(1 - ||\mathbf{v}||^2)} < \frac{5}{4}.$$

Substituting in the expressions for $\mathbf{u}$ and $\mathbf{v}$ gives

$$\frac{(\lambda - v_1)^2 + v_2^2}{(1 - \lambda^2)(1 - v_1^2 - v_2^2)} < \frac{1}{8}$$

and since the two terms in the denominator are positive,

$$\begin{aligned}
8v_1^2 - 16\lambda v_1 + 8\lambda^2 + 8v_2^2 &< (1 - \lambda^2) - (v_1^2 + v_2^2)(1 - \lambda^2) \\
(9 - \lambda^2)v_1^2 - 16\lambda v_1 + (9 - \lambda^2)v_2^2 &< 1 - 9\lambda^2 \\
(9 - \lambda^2)\left(v_1 - \frac{8\lambda}{9 - \lambda^2}\right)^2 - \frac{64\lambda^2}{9 - \lambda^2} + (9 - \lambda^2)v_2^2 &< 1 - 9\lambda^2 \\
\left(v_1 - \frac{8\lambda}{9 - \lambda^2}\right)^2 + v_2^2 &< \frac{(1 - 9\lambda^2)(9 - \lambda^2) + 64\lambda^2}{(9 - \lambda^2)^2} \\
\left(v_1 - \frac{8\lambda}{9 - \lambda^2}\right)^2 + v_2^2 &< \frac{9(1 - \lambda^2)^2}{(9 - \lambda^2)^2} = \left(\frac{3(1 - \lambda^2)}{9 - \lambda^2}\right)^2.
\end{aligned}$$

This defines a disk, centered on $(8\lambda/(9-\lambda^2), 0)$, with radius $3(1-\lambda^2)/(9-\lambda^2)$. Hence, the three neighborhoods are

- $N_r((0,0))$: a disk, radius $1/3$, centered on $(0,0)$,
- $N_r((1/2,0))$: a disk, radius $9/35$, centered on $(16/35, 0)$,
- $N_r((3/4,0))$: a disk, radius $7/45$, centered on $(32/45, 0)$.

Figure 1(a) shows a picture of these neighborhoods. Even though the correspond to the same radius, the circles become progressively smaller for points further away from the origin, corresponding to distances becoming larger in the Poincaré metric.

Another approach to compute the neighborhoods is to make use of a small computer program to scan the disk, and directly test whether a point is within the disk or not. A C++ program to do this is given in the appendix. The neighborhoods, shown in Fig. 1(b), agree with those computed analytically.

5. Write $N_r^{(i)}(p)$ for the neighborhood of radius r at p with respect to the metric d_i. If the two metrics are equivalent, then for all $p \in S$, and for all $\epsilon > 0$, there exists a $\delta > 0$ such that
$$N_\delta^{(2)}(p) \subseteq N_\epsilon^{(1)}(p).$$
In addition, for all $\in S$, and for all $\epsilon > 0$, there exists a $\delta > 0$ such that
$$N_\delta^{(1)}(p) \subseteq N_\epsilon^{(2)}(p).$$
Now consider a sequence (s_n) that converges to s with respect to d_1. Choose any $\epsilon > 0$. Then there exists a $\delta > 0$ such that $N_\delta^{(1)}(p) \subseteq N_\epsilon^{(2)}(p)$. Since s_n converges to s with respect to d_1, there exists and N such that $n > N$ implies
$$d_1(s_n, s) < \delta$$
and thus $s_n \in N_\delta^{(1)}(s)$. Hence for all $n > N$, $s_n \in N_\epsilon^{(2)}(s)$, and thus $d_2(s_n, s) < \epsilon$. Hence (s_n) converges to s with repsect to d_2 also.

6. Choose an $\epsilon > 0$. Then there exists a K such that $k > K$ implies that
$$d(p_{n_k}, p) < \frac{\epsilon}{2}.$$
Since (p_n) is a Cauchy sequence, there exists N such that for all $n, m > N$,
$$d(p_m, p_n) < \frac{\epsilon}{2}.$$
Now, since there are an infinite number of available terms, there exists an index n_l in the subsequence such that $l > K$, and $n_l > N$. Now, if $n > N$, then by using the triangle inequality,
$$\begin{aligned} d(p_n, p) &\leq d(p_n, p_{n_l}) + d(p_{n_l}, p) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus the full sequence (p_n) converges to p.

7. Suppose that (p_n) and (q_n) are Cauchy sequences in a space X with metric d. From the triangle inequality,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

for all m and n, and hence

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n).$$

Swapping n and m gives

$$d(p_m, q_m) - d(p_n, q_n) \leq d(p_n, p_m) + d(q_m, q_n)$$

and combining with the previous expression shows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n).$$

Now consider the sequence (a_n) defined by $a_n = d(p_n, q_n)$. Choose $\epsilon > 0$. Then there exists an N_1, such that for all $m, n > N_1$, $d(p_n, p_m) < \epsilon/2$. Similarly, there exists an N_2, such that for all $m, n > N_2$, $d(q_n, q_m) < \epsilon/2$. Define $N = \max\{N_1, N_2\}$. Then for all $m, n > N$,

$$\begin{aligned} |a_m - a_n| &= |d(p_n, q_n) - d(p_m, q_m)| \\ &\leq d(p_n, p_m) + d(q_m, q_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence (a_n) is a Cauchy sequence. Since the $a_n \in \mathbb{R}$, this implies that (a_n) converges.

8. This problem can be most easily done by making use of some linear algebra. For $\mathbf{u} = (u_1, u_2)$, the alternative norm can be written as

$$\begin{aligned} ||\mathbf{u}||_S &= \sqrt{u_1^2 + u_2^2 + u_1 u_2} \\ &= \sqrt{\left(u_1 + \frac{u_2}{2}\right)^2 + \frac{3u_2^2}{4}} \\ &= \sqrt{w_1^2 + w_2^2} \\ &= ||\mathbf{w}|| \end{aligned}$$

for $\mathbf{w} = A\mathbf{u}$, where A is the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$$

and $||\mathbf{w}||$ refers to the Euclidean norm of $\mathbf{w}$. Hence, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, the alternative metric can be written in terms of the Euclidean metric as

$$\begin{aligned} d_S(\mathbf{u}, \mathbf{v}) &= ||\mathbf{u} - \mathbf{v}||_S \\ &= ||A(\mathbf{u} - \mathbf{v})|| \\ &= d_E(A\mathbf{u}, A\mathbf{v}). \end{aligned}$$

The metric A is invertible, with inverse

$$A^{-1} = \begin{pmatrix} 1 & 1/\sqrt{3} \\ 0 & 2/\sqrt{3} \end{pmatrix}.$$

Checking that d_S is a metric can then be carried out by making use of the fact that d_E is a metric:

M1. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, $d_S(\mathbf{u}, \mathbf{v}) = d_E(A\mathbf{u}, A\mathbf{v}) \geq 0$. Since the matrix A is invertible, $\mathbf{u} = \mathbf{v}$ if and only if $A\mathbf{u} = A\mathbf{v}$, and thus $d_S(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.

M2. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, $d_S(\mathbf{u}, \mathbf{v}) = d_E(A\mathbf{u}, A\mathbf{v}) = d_E(A\mathbf{v}, A\mathbf{u}) = d_S(\mathbf{v}, \mathbf{u})$ and thus the metric is symmetric.

M3. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^2$,

$$\begin{aligned} d_S(\mathbf{u}, \mathbf{w}) &= d_E(A\mathbf{u}, A\mathbf{w}) \\ &\leq d_E(A\mathbf{u}, A\mathbf{v}) + d_E(A\mathbf{v}, A\mathbf{w}) \\ &= d_S(\mathbf{u}, \mathbf{v}) + d_S(\mathbf{v}, \mathbf{w}) \end{aligned}$$

and thus the triangle inequality is satisfied.

To show that the two metrics are equivalent, consider any $\epsilon > 0$. Since both the Euclidean norm and the alternative norm are invariant under translations, it suffices to prove equivalence at $\mathbf{0} = (0, 0)$. If $\mathbf{u} \in N^E_{\epsilon/2}(\mathbf{0})$, then

$$u_1^2 + u_2^2 < \frac{\epsilon^2}{4}$$

and thus $|u_1| < \epsilon/2$ and $|u_2| < \epsilon/2$, so $u_1 < \epsilon/2$ and $u_2 < \epsilon/2$. Hence

$$u_1^2 + u_2^2 + u_1 u_2 < \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2.$$

Thus $d_S(\mathbf{u}) < \epsilon$, and hence $\mathbf{u} \in N^S_\epsilon(\mathbf{0})$. Hence $N^E_{\epsilon/2}(\mathbf{0}) \subseteq N^S_\epsilon(\mathbf{0})$. Now suppose that $\mathbf{u} \in N^S_{\epsilon/2}(\mathbf{0})$. Then

$$\left(u_1 + \frac{u_2}{2}\right)^2 + \frac{3u_2^2}{4} < \frac{\epsilon^2}{4}$$

and hence

$$\left| u_1 + \frac{u_2}{2} \right| < \frac{\epsilon}{2}, \qquad \frac{\sqrt{3}}{2} |u_2| < \frac{\epsilon}{2}.$$

By making use of the triangle inequality,

$$|u_1| < \frac{\epsilon}{2} + \frac{|u_2|}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2\sqrt{3}} = \frac{\epsilon}{2} \left(\frac{\sqrt{3}+2}{2\sqrt{3}} \right).$$

Hence

$$u_1^2 + u_2^2 < \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} \left(\frac{7+4\sqrt{3}}{6} \right) < \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} \left(\frac{7+4 \times 2}{6} \right) < \epsilon^2$$

and thus $\mathbf{u} \in N^E_\epsilon(\mathbf{0})$. Hence $N^S_{\epsilon/2}(\mathbf{0}) \subseteq N^E_\epsilon(\mathbf{0})$ and the two metrics are equivalent.

Appendix

This short C++ code scans points in the Poincaré disk to see whether they are within the neighborhoods of interest. The points found to be in each neighborhood are saved into text files, which can be read by many plotting programs.

```
#include <cstdio>
using namespace std;

// Constants setting the size and spacing of the scanning grid
const double h(0.02);
const int M(50);

// This function returns the hyperbolic cosh of the metric. There's no need to
// do inverse cosh, since the radius is defined in terms of an inverse cosh
// anyway.
double cosh_d(double u1,double u2,double v1,double v2) {
        return 1+2*((u1-v1)*(u1-v1)+(u2-v2)*(u2-v2))
                /((1-(u1*u1+u2*u2))*(1-(v1*v1+v2*v2)));
}

int main() {
        double x,y;

        // Open output files for each of the three neighborhoods
        FILE *F1(fopen("N1","w")),
             *F2(fopen("N2","w")),
             *F3(fopen("N3","w"));

        // Check that output files opened correctly
        if(F1==NULL||F2==NULL||F3==NULL) {
                fputs("File open error\n",stderr);
                return 1;
        }

        // Sweep through points in a square
        for(y=-M*h;y<(0.5+M)*h;y+=h) {
                for(x=-M*h;x<(0.5+M)*h;x+=h) {

                        // Skip points that lie outside the disk
                        if(x*x+y*y>=1) continue;

                        // Check whether the current point is within any of the
                        // neighborhoods. If so, then save this point to the
                        // corresponding output file.
                        if(cosh_d(x,y,0,0)<1.25) fprintf(F1,"%g %g\n",x,y);
                        if(cosh_d(x,y,0.5,0)<1.25) fprintf(F2,"%g %g\n",x,y);
                        if(cosh_d(x,y,0.75,0)<1.25) fprintf(F3,"%g %g\n",x,y);
                }
        }

        // Close the output files
        fclose(F1);
        fclose(F2);
        fclose(F3);
}
```