Math 104: Homework 4 solutions

1. (a) From the diagram, it can be seen that for every $m \in \mathbb{N}$, the fraction $\frac{1}{m}$ occurs infinitely many times in the sequence, and thus there is a constant subsequence $s_{n_k} = \frac{1}{m}$, so $\frac{1}{m} \in S$. In addition there is a subsequence $t_k = s_{n_k} = \frac{1}{k}$ which converges to 0, so $0 \in S$.

   Now consider any number $l < 0$. For all $n \in \mathbb{N}$, $|s_n - l| > |l|$ and thus $l \notin S$. Similarly, consider any number $l > 1$. Then for all $n \in \mathbb{N}$, $|s_n - l| > l - 1$ and thus $l \notin S$. Finally, suppose $0 < l < 1$ but $l \notin \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, there exists an $m \in \mathbb{N}$ such that $\frac{1}{m+1} < l < \frac{1}{m}$, and hence for all $n \in \mathbb{N}$, $|s_n - l| > \min\{\frac{1}{m} - l, l - \frac{1}{m+1}\} > 0$, so $l \notin S$.

   Therefore $S = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

   (b) By Theorem 11.7, $\limsup s_n = \sup S = 1$ and $\liminf s_n = \inf S = 0$.

2. Suppose that $|s_n|$ is bounded. Then there exists an M such that $|s_n| < M$ for all n. Hence for all N, $v_N = \sup\{|s_n| : n > N\} \leq M$, since M is an upper bound for the set of all $|s_n|$. Since v_N is bounded and monotonic it must converge to a real number and thus $\limsup |s_n| < \infty$.

   Now suppose the $\limsup |s_n| < \infty$. Since $|s_n| \geq 0$ for all n, it is not possible for $\limsup |s_n| = -\infty$, and thus $\limsup |s_n| = m$ for some real number m. By Theorem 9.1 convergent sequences are bounded, and thus $v_N < M$ for all $N \in \mathbb{N}$ and for some $M \in \mathbb{R}$. Hence $v_1 = \sup\{|s_n| : n > 1\} < M$. Hence $|s_n| \leq M$ for all $n \geq 2$, and thus $|s_n| \leq \max\{s_1, M\}$ for all $n \in \mathbb{N}$. Hence (s_n) is bounded.

3. Suppose (a_n) is a sequence defined for $n \in \mathbb{N}$ such that $\liminf |a_n| = 0$. Now define a subsequence (a_{n_k}) where $n_1 < n_2 < n_3 < \ldots$ are defined such that

$$|a_{n_k}| \leq \frac{1}{k^2}$$

   for all $k \in \mathbb{N}$. In addition, set $n_0 = 0$ for convenience.

   To show it always possible to choose a sequence this way, suppose it was not possible to choose an n_k for some $k \in \mathbb{N}$. In that case, for all $n > n_{k-1}$, $|a_n| > 1/k^2$. Then $\inf\{|a_n| : n > n_{k-1}\} \geq 1/k^2$, which implies that $\liminf |a_n| \geq 1/k^2$ which would be a contradiction.

   It has been previously shown that $\sum_{k=1}^{\infty} 1/k^2$ converges. Hence, by using the comparison test with the above inequality, $\sum_{k=1}^{\infty} a_{n_k}$ converges also.

4. (a) Define $a_n = n^3/2^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{2^{n+1}} \frac{2^n}{n^3} = \frac{(1 + \frac{1}{n})^3}{2}.$$

Since $\frac{1}{n} \to 0$ as $n \to \infty$, then by the limit theorems for addition and multiplication of sequences, $a_{n+1}/a_n \to 1/2$. Hence, by the ratio test, $\sum n^3/2^n$ converges.

(b) Consider a partial sum of the first N terms:

$$\sum_{n=1}^{N} \sqrt{n+1} - \sqrt{n} = (\sqrt{N+1} - \sqrt{N}) + (\sqrt{N} - \sqrt{N-1}) + \ldots + (\sqrt{2} - \sqrt{1}).$$

Since all but the first and last terms cancel,

$$\sum_{n=1}^{N} \sqrt{n+1} - \sqrt{n} = \sqrt{N+1} - \sqrt{1} = \sqrt{N+1} - 1,$$

and thus $\sum \sqrt{n+1} - \sqrt{n}$ diverges to infinity.

(c) Define $a_n = 1/\sqrt{n!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \frac{1}{\sqrt{(n+1)}}$$

and hence $a_{n+1}/a_n \to 0$ as $n \to \infty$. Hence, by the ratio test, $\sum 1/\sqrt{n!} \to 0$.

(d) Define $a_n = 2^{-3n+(-1)^n}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{-3(n+1)+(-1)^{n+1}}}{2^{-3n+(-1)^n}} = 2^{-3+(-1)^{n+1}-(-1)^n} = 2^{-3-2(-1)^n},$$

which is $1/2$ for even n and $1/32$ for odd n. Hence $\limsup |a_{n+1}|/|a_n| = 1/2$, and the series converges.

(e) Define $a_n = n!/n^n$, and let $b_n = 2/n^2$. Observe that $a_1 = 1/1 \leq b_1$. For $n \geq 2$,

$$\begin{aligned} \frac{n!}{n^n} &= \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) \\ &\leq \left(\frac{1}{n}\right)\left(\frac{2}{n}\right) = b_n. \end{aligned}$$

Hence, since $0 < a_n \leq b_n$ for all n, and $\sum b_n$ converges, then $\sum a_n$ converges.

5. (a) If (u_n) and (v_n) are equal apart from at finitely many n, then there exists an $N_1 \in \mathbb{N}$ such that $u_n = v_n$ for all $n > N_1$. Now, pick any $\epsilon > 0$, and suppose $\sum u_n$ converges. Then there exists an $N_2 \in \mathbb{N}$ such that $n \geq m > N_2$ implies that

$$\left| \sum_{k=m}^{n} u_k \right| < \epsilon.$$

Set $N = \max\{N_1, N_2\}$. Then for all $n \geq m > N$,

$$\left|\sum_{k=m}^{n} v_k\right| = \left|\sum_{k=m}^{n} u_k\right| < \epsilon,$$

and hence $\sum v_n$ converges. The same argument can be used to show that if $\sum v_n$ converges, then so does $\sum u_n$. Hence $\sum u_n$ and $\sum v_n$ either both converge or both diverge.

(b) This is false. Let

$$u_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2}{n} & \text{if } n \text{ is even} \end{cases}$$

and

$$v_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{4}{n^2} & \text{if } n \text{ is even.} \end{cases}$$

Hence, $u_n = v_n$ for all odd n. However, if $n = 2N$ or $n = 2N + 1$, then

$$\sum_{k=1}^{n} u_k = \sum_{k=1}^{N} \frac{1}{k},$$

which diverges, but

$$\sum_{k=1}^{n} v_k = \sum_{k=1}^{N} \frac{1}{k^2}$$

which converges.

(c) If $(u_n/v_n) \to 1$ as $n \to \infty$, then there exists an N such that $n > N$ implies that

$$\left|\frac{u_n}{v_n} - 1\right| < \frac{1}{2}$$

and hence

$$\frac{1}{2} < \frac{u_n}{v_n} < \frac{3}{2},$$

so that

$$\frac{v_n}{2} < u_n < \frac{3v_n}{2}.$$

Now suppose $\sum v_n$ converges. Since $\sum_{n=N}^{\infty} \frac{3v_n}{2}$ converges, then $\sum_{n=N}^{\infty} u_n$ converges by the comparison test. Hence $\sum_{n=1}^{\infty} u_n$ converges, since it differs by a finite sum of terms.

Suppose $\sum u_n$ converges. Since $v_n \leq 2u_n$ for all $n > N$, then $\sum_{n=N}^{\infty} v_n$ converges by the comparison test. Hence $\sum_{n=1}^{\infty} v_n$ converges, since it differs by a finite sum of terms. Hence $\sum u_n$ and $\sum v_n$ either both converge or both diverge.

(d) This is false. Consider $u_n = 1/n$ and $v_n = 1/n^2$. Then since $u_n \to 0$ and $v_n \to 0$, the limit theorems for sequences assert that $u_n - v_n \to 0$. However, $\sum u_n$ diverges but $\sum v_n$ converges.

(e) This is false. Consider the sequence $u_n = 2^{(-1)^n - n}$. By the root test,

$$(u_n)^{1/n} = 2^{((-1)^n - n)/n} \to 1/2$$

as $n \to \infty$, and thus the sequence is convergent. However, if n is odd,

$$\frac{u_{n+1}}{u_n} = \frac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n+(-1)^n}} = \frac{2^{-n-1+1}}{2^{-n-1}} = 2.$$

Thus, there are an infinite number of terms where $u_{n+1}/u_n > 3/2 > 1$.

6. Consider the sequence $a_n = (-1)^n$. Then

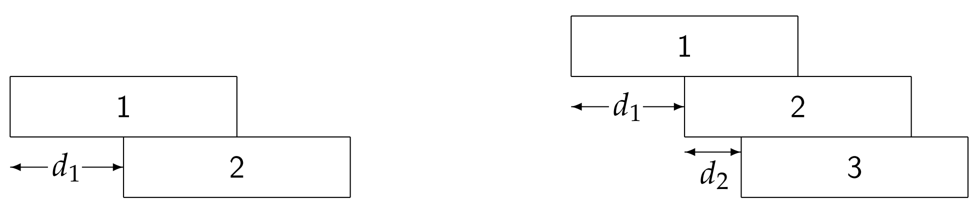
$$\sum_{n=1}^{2N} = (-1) + 1 + \ldots + (-1) + 1 = 0,$$

and thus $\sum_{n=1}^{2N} a_n \to 0$ as $N \to \infty$. Similarly,

$$\sum_{n=1}^{2N+1} = (-1) + 1 + \ldots + (-1) + 1 + (-1) = -1.$$

and thus $\sum_{n=1}^{2N} a_n \to -1$ as $N \to \infty$. However, $\sum a_n$ alternates between values of 0 and -1, and thus there is no N' such that $n > N'$ implies $|\sum_{k=1}^{n} a_k - l| < 1/2$. Hence $\sum a_n$ is divergent.

7. Since bricks are uniform, their center of mass is a distance $1/2$ from one end.



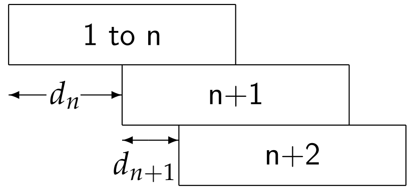
(a) (b)

In diagram (a), the bricks will be stable as long as the center of mass of brick 1 does not overhang the edge of brick 2, and thus $d_1 = 1/2$.

In diagram (b), the bricks will be stable as long as the combined center of mass of bricks 1 and 2 does not edge of brick 3. The center of mass is a distance

$$\frac{0 + 1/2}{2} = 1/4$$

from the left side of brick 2, and thus $d_2 = 1/4$. The aim now is to show that in general, $d_n = \frac{1}{2n}$. Suppose the result is true for n and consider the case for $n + 1$.



If the result is true for n, then as shown in the diagram, bricks 1 to n can be thought of as being replaced by one brick of mass n. Then the combined center of mass of this brick and brick n+1 is a distance

$$\frac{0 \cdot n + 1/2 \cdot 1}{n+1} = \frac{1}{2(n+1)}$$

from the left side of brick n+1, and thus $d_{n+1} = \frac{1}{2(n+1)}$. Hence by mathematical induction, $d_n = \frac{1}{2n}$ for all $n \in \mathbb{N}$.

Since $\sum \frac{1}{n}$ diverges, then $\sum \frac{1}{2n}$ diverges also. Amazingly, this means that it is possible to make a free-standing tower of bricks with an arbitrarily large overhang. Even in practice, it is possible to build a tower of bricks where $\sum d_n$ is bigger than one or two.