Math 104: Homework 3 solutions

1. Suppose t_n is bounded so that $|t_n| \leq M$ for all n, where $M \geq 0$. If $M = 0$, then $|t_n| \leq 0$ for all n implies that $t_n = 0$ for all n. Thus $s_n t_n = 0$ for all n and hence $\lim_{n\to\infty} s_n t_n = 0$.

   Otherwise $M > 0$. Consider any $\epsilon > 0$. Since s_n converges to zero, there exists an N such that for all $n > N$,
   $$|s_n - 0| = |s_n| < \frac{\epsilon}{M}.$$
   Hence for all $n > N$,
   $$|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| \leq |s_n|M < \frac{\epsilon}{M}M = \epsilon.$$
   Thus, for all $\epsilon > 0$, there exists an N such that $n > N$ implies that $|s_n t_n - 0| < \epsilon$, and thus $\lim_{n\to\infty} s_n t_n = 0$.

2. (a) Consider any $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = s$, there exists an N_1 such that $n > N_1$ implies
   $$|a_n - s| < \epsilon$$
   and hence
   $$a_n > s - \epsilon. \tag{1}$$
   Similarly, since $\lim_{n\to\infty} b_n = s$, there exists an N_2 such that $n > N_2$ implies
   $$|b_n - s| < \epsilon$$
   and thus
   $$b_n < s + \epsilon. \tag{2}$$
   Define $N_3 = \max\{N_1, N_2\}$. If $n > N_3$, then both of the conditions given in Eqs. 1 and 2 will hold. Since $a_n \leq s_n \leq b_n$ it follows that
   $$s - \epsilon < s_n < s + \epsilon.$$
   and thus
   $$-\epsilon < s - s_n < \epsilon.$$
   The result from Exercise 3.5(a) shows that this statement is equivalent to $|s - s_n| < \epsilon$. Hence, for all $\epsilon > 0$, it is possible to construct an N_3 such that $n > N_3$ implies $|s - s_n| < \epsilon$, so $\lim_{n\to\infty} s_n = s$.

   (b) If $|s_n| \leq t_n$ for all n, it implies that $-t_n \leq s_n \leq t_n$ for all n. Since $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} -t_n = 0$, the result from part (a) can be applied and thus $\lim_{n\to\infty} s_n = 0$.

3. (a) Suppose that $a = \lim x_n$. Then, by using the limit theorems for addition and multiplication,

$$a = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 3x_n^2 = 3(\lim_{n\to\infty} x_n)^2 = 3a^2.$$

Thus a would have to satisfy the quadratic equation $a = 3a^2$, and hence $a = 0$ or $a = 1/3$.

(b) By induction, it can be shown that $x_n \geq n$ for all $n \geq 1$. For $n = 1$, $x_1 = 1 \geq 1$ which is satisfied. Now suppose that the result is true for n and consider the case for $n + 1$:

$$x_{n+1} = 3x_n^2 \geq 3n^2 \geq n^2 + 1 \geq n + 1,$$

where we have used the result that $n^2 \geq n$ for $n \geq 1$. Hence the induction step is true, and by mathematical induction, $x_n \geq n$ for all n. Now for any $M > 0$, then for all $n > M$, $x_n > M$ and thus (x_n) diverges to ∞.

(c) From part (b), it is clear that (x_n) does not converge to a finite limit, and thus the assumption used in part (a) is not valid. Hence, the result of (a) does not provide any information about how the actual sequence behaves.

For different values of x_1, different behavior is observed, and part (a) is useful. Suppose $0 < x_1 < 1/3$. Then it can be shown that $\lim x_n = 0$. If $x_1 = 1/3$ then $x_n = 1/3$ for all n and $\lim x_n = 1/3$. However if $x_1 = 1/3 + \epsilon$ where $\epsilon \neq 0$ and small, then $x_2 \approx 1/3 + 2\epsilon$, meaning that it progressively moves further away from $1/3$. This is frequently refered to as an unstable equilibrium.

4. Figure 1 shows a plot of the function $f(x) = 2x/(1 + x)$, with web plots for two different values of t. For both $t < 1$ and $t > 1$, the sequences rapidly converge to 1. Define a sequence (d_n) according to $a_n = 1 + d_n$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} a_{n+1} &= \frac{2a_n}{1 + a_n} \\ &= \frac{2 + 2d_n}{2 + d_n} \\ &= 1 + \frac{d_n}{2 + d_n} \end{aligned}$$

and hence $d_{n+1} = d_n/(2 + d_n)$. Consider two cases. First, if $0 < t < 1$, then $-1 < d_1 < 0$. Then define $\Delta = 1/(2 + d_1)$, and observe that $1/2 < \Delta < 1$. Now, $d_2 = d_1\Delta$, so $-1 < d_1 < d_2 < 0$.

Now, suppose that $d_1 < d_n < 0$. Then $d_{n+1} = d_n/(2 + d_n) \geq d_n\Delta$, and hence $d_1 < d_{n+1} < 0$. Hence, by mathematical induction $d_1 < d_n < 1$ for all $n \in \mathbb{N}$, and $d_{n+1} \geq d_n\Delta$ for all $n \in \mathbb{N}$. Hence

$$d_1\Delta^{n-1} \leq d_n < 0.$$

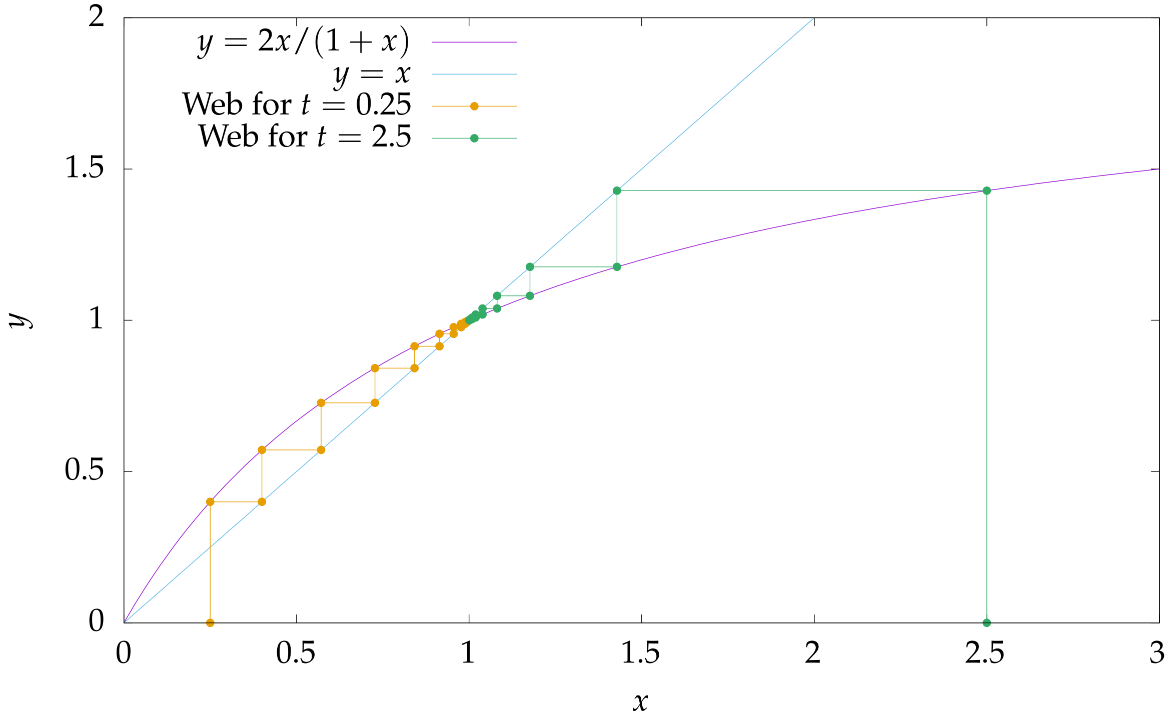

Figure 1: A plot of the function that is used to define the recursive sequence in question 3. Web plots are shown for two initial values of t.

Since $\Delta^{n-1} \to 0$ as $n \to \infty$ by Theorem 9.7(c), then $d_n \to 0$ by the Squeezing Lemma. Hence $a_n \to 1$.

The second case is $t \geq 1$, which corresponds to $d_1 \geq 0$. If $d_n \geq 0$, then $0 \leq d_{n+1} \leq d_n/2$. Hence $0 \leq d_n \leq d_1({}^1\!/\!{}_2)^{n-1}$ for all $n \in \mathbb{N}$. Since $({}^1\!/\!{}_2)^n \to 0$, then $d_n \to 0$ by the Squeezing Lemma and hence $a_n \to 1$.

5. This problem can be broken into two parts. First, suppose that (s_n) is a Cauchy sequence, and consider (c_n) defined according to $c_n = ks_n$ for all n and for some $k \in \mathbb{R}$. Suppose $k = 0$, so that $c_n = 0$ is a constant sequence. Then for all $\epsilon > 0$, $|c_n - c_m| = 0$ for all m and n, and thus (c_n) is a Cauchy sequence. Now consider $k \neq 0$, and choose $\epsilon > 0$. Then, since (s_n) is a Cauchy sequence, there exists an N such that for all $m, n > N$,
$$|s_n - s_m| < \frac{\epsilon}{|k|}.$$

Hence for all $m, n > N$,

$$\begin{aligned} |c_n - c_m| &= |ks_n - ks_m| \\ &= |k| \cdot |s_n - s_m| \\ &< |k| \frac{\epsilon}{|k|} = \epsilon \end{aligned}$$

and thus (c_n) is a Cauchy sequence.

Now suppose that (c_n) and (d_n) are Cauchy sequences, and consider (u_n) defined by $u_n = c_n + d_n$. Choose any $\epsilon > 0$. Since (c_n) is a Cauchy sequence, there exists an N_1 such that for all $m, n > N_1$,

$$|c_n - c_m| < \frac{\epsilon}{2}.$$

Similarly, there exists an N_2 such that for all $m, n > N_2$,

$$|d_n - d_m| < \frac{\epsilon}{2}.$$

Now set $N = \max\{N_1, N_2\}$. Then for all $m, n > N$,

$$\begin{aligned} |u_n - u_m| &= |(c_n + d_n) - (c_m + d_m)| \\ &= |(c_n - c_m) + (d_n - d_m)| \\ &\leq |c_n - c_m| + |d_n - d_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence (u_n) is a Cauchy sequence.

Now suppose (u_n) is a sequence defined as $u_n = as_n + bt_n$ for $a, b \in \mathbb{R}$, where (s_n) and (t_n) are Cauchy sequences. By the first result, if (c_n) and (d_n) are sequences satisfying $c_n = as_n$ and $d_n = bt_n$, then they are Cauchy sequences. Since $u_n = c_n + d_n$, then by the second result, it is a Cauchy sequence also.

6. (a) Pick any $M > 0$. Since (s_n) diverges to ∞, there exists an N such that for all $n > N$,

$$s_n > \frac{M}{k}.$$

Hence for all $n > N$,

$$ks_n > k\frac{M}{k} = M$$

and thus $\lim ks_n = \infty$.

(b) Suppose (s_n) diverges to ∞. Then for any $M < 0$ there exists an N such that for all $n > N$,

$$s_n > -M.$$

Hence, by the ordering axioms,

$$-s_n < M$$

for all $n > N$, and thus $\lim(-s_n) = -\infty$.

Now suppose $\lim(-s_n) = -\infty$. Then for any $M > 0$, there exists an N such that for all $n > N$, $-s_n < -M$ and hence $s_n > M$. Hence $\lim s_n = \infty$ if and only if $\lim(-s_n) = \infty$.

(c) If $k < 0$, then $-k > 0$. By part (a), if $\lim s_n = \infty$, then $\lim(-k)s_n = \infty$. By part (b), this implies that $\lim ks_n = -\infty$.

7. The $(n+1)$th term can be divided into n fractions of size $\frac{1}{n}$, and written as

$$s_{n+1} = \frac{1}{n}(s_{n+1} + s_{n+1} + \ldots s_{n+1}).$$

Since (s_n) non-decreasing sequence, $s_{n+1} \geq s_k$ for $k = 1, 2, \ldots, n$ and hence

$$s_{n+1} \geq \frac{1}{n}(s_1 + s_2 + \ldots + s_n).$$

Now consider the $(n+1)$th average:

$$\begin{aligned}
\sigma_{n+1} &= \frac{1}{n+1}(s_1 + s_2 + \ldots + s_n + s_{n+1}) \\
&\geq \frac{1}{n+1}\left(s_1 + s_2 + \ldots + s_n + \frac{1}{n}(s_1 + s_2 + \ldots + s_n)\right) \\
&\geq \frac{1}{n+1}\frac{n+1}{n}(s_1 + s_2 + \ldots + s_n) \\
&\geq \frac{1}{n}(s_1 + s_2 + \ldots + s_n) \\
&\geq \sigma_n,
\end{aligned}$$

and thus this sequence is nondecreasing.

8. Consider the set $S_N = \{s_n \mid n > N\}$ for any $N \in \mathbb{N}$. Then $-1 \in S_N$ since there exists an even number $k > N$ and $s_k = -1$. Since $1 + \frac{1}{n} > -1$ for all n, it follows that $\inf S_N = \min S_N = -1$. Thus $u_N = -1$ for all $N \in \mathbb{N}$.

For any N in $\mathbb{N}$, define L to be the smallest odd number satisfying $L > N$. This can be explicitly defined as

$$L = \begin{cases} N+1 & \text{if } N \text{ is even,} \\ N+2 & \text{if } N \text{ is odd.} \end{cases}$$

Then $1 + \frac{1}{L} \in S_N$. Consider any $n > N$. If n is odd, then $s_n = 1 + \frac{1}{n}$ and since $n \geq L$ it follows that $s_n \leq 1 + \frac{1}{L}$. If n is even, then $s_n = -1$ and thus $s_n \leq 1 + \frac{1}{L}$ also. Thus $\sup S_N = \max S_N = 1 + \frac{1}{L}$. Hence

$$v_N = 1 + \frac{1}{L} = \begin{cases} 1 + \frac{1}{N+1} & \text{if } N \text{ is even,} \\ 1 + \frac{1}{N+2} & \text{if } N \text{ is odd.} \end{cases}$$

Since u_N is a constant sequence, it follows that

$$\liminf s_n = \lim_{N \to \infty} u_N = -1.$$

To determine the convergence of v_N, Exercise 8.5(a) can be employed. Define $a_N = 1$ and $b_N = 1 + \frac{1}{N}$. Then $a_N \leq v_N \leq b_N$ for all N, and $\lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N = 1$, so

$$\limsup s_n = \lim_{N \to \infty} v_N = 1.$$

Note that in this case $\liminf s_n \neq \limsup s_n$ and thus by Theorem 10.7 it follows that $\lim_{n \to \infty} s_n$ is undefined.