Math 104: Homework 2 solutions

1. - $A = (0, \infty)$: Since this is an open interval, the minimum is undefined, and since the set is not bounded above, the maximum is also undefined. $\inf A = 0$ and $\sup A = \infty$.
   - $B = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$: This set does not have a minimum, since for any element $\frac{1}{m} + \frac{1}{n}$, there is a smaller element $\frac{1}{m+1} + \frac{1}{n}$. The maximum element is 2, which is attained for $m = n = 1$. Hence $\max B = 2$. B is a bounded below by 0. However, for $\epsilon > 0$, there exists $c \in N$ such that $1/c < \epsilon$, by the Archimedean property. Thus by putting $m = n = 2c$, we see that there exists $b \in B$ such that $b < \epsilon$. Hence, ϵ is not a lower bound. Hence 0 is the greatest lower bound, and thus $\inf B = 0$.
   - $C = \{x^2 - x - 1 : x \in \mathbb{R}\}$: By completing the square, this can be written as $\{(x - \frac{1}{2})^2 - \frac{5}{4} | x \in \mathbb{R}\}$. We know that $(x - \frac{1}{2})^2 \geq 0$ by Theorem 3.2(iv). Hence $\min C = -\frac{5}{4}$ which is attained for $x = \frac{1}{2}$. Since the values of C are not bounded above, the maximum does not exist. Hence $\inf C = -\frac{5}{4}$ and $\sup C = \infty$.
   - $D = [0, 1] \cup [2, 3]$: Since this set is composed of closed intervals, we have $\min D = 0$ and $\max D = 3$. Hence $\inf D = 0$ and $\max D = 3$, and thus $\inf D = 0$ and $\sup D = 3$.
   - $E = \cup_{n=1}^{\infty} [2n, 2n + 1]$: The first interval in this union is $[2, 3]$, and there are an infinite number of consecutive intervals in the positive direction. Hence $\min E = 2$, but the maximum does not exist. Thus $\inf E = 2$ and $\sup E = \infty$.
   - $F = \cap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$: We begin by showing that $F = \{1\}$. Choose any $x > 1$. Then $x = 1 + \epsilon$ for $\epsilon > 0$. Hence, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Hence $x \notin (1 - \frac{1}{n}, 1 + \frac{1}{n})$, and thus $x \notin F$. Similarly if $x < 1$, then $x = 1 - \epsilon$, and and there exists an n such that $x \notin (1 - \frac{1}{n}, 1 + \frac{1}{n})$. However, $1 \in (1 - \frac{1}{n}, 1 + \frac{1}{n})$ for all $n \in \mathbb{N}$. Thus $F = \{1\}$, and hence $\min F = \max F = \inf F = \sup F = 1$.

2. (a) To begin, we show that $\sup A + \sup B$ is an upper bound for S. Any element in S can be written as $a + b$ for $a \in A$, and $b \in B$. However, since $\sup A$ is an upper bound for A, then $a \leq \sup A$. Similiarly, $b \leq \sup B$, and thus $a + b \leq \sup A + \sup B$.

   We now wish to show that $\sup A + \sup B$ is the least upper bound for S. Assume that t is a upper bound for S, but that $t < \sup A + \sup B$. Then for some $\epsilon > 0$, $t = \sup A + \sup B - \epsilon$. Now, since $\sup A$ is the supremum of A, there exists $a \in A$ such that $a > \sup A - \frac{\epsilon}{2}$. (If this was not the case, then $\sup A - \frac{\epsilon}{2}$ would be an upper bound for A.) Similarly, there exists $b \in B$ such that $b > \sup B - \frac{\epsilon}{2}$. But $a + b \in S$, and

$$a + b > \left(\sup A - \frac{\epsilon}{2}\right) + \left(\sup B - \frac{\epsilon}{2}\right) = t.$$

Hence t is not an upper bound, which is a contradiction. Thus if t is an upper bound, it must satisfy $t \geq \sup A + \sup B$.

$\sup A + \sup B$ is an upper bound for S, and it is the least upper bound. Hence $\sup S = \sup A + \sup B$.

(b) This could be proved by repeating the above argument but with lower bounds instead of upper bounds. However, an alternative method is to define negated sets $-A = \{-a | a \in A\}$, $-B = \{-b | b \in B\}$, and $-S = \{-s | s \in S\}$.

We see that $-S$ can be constructed as the set of sums $a' + b'$ where $a' \in -A$ and $b' \in -B$. Thus, by applying the above result, we know that $\sup(-S) = \sup(-A) + \sup(-B)$. However, by Corollary 4.5, for any set C, $\sup(-C) = -\inf C$. Hence $-\inf S = -\inf A - \inf B$ and thus $\inf S = \inf A + \inf B$.

3. This result is not true. As a counterexample, choose $A = B = \{-2, 1\}$. Then $\sup A = \sup B = 1$, and hence $\sup A \cdot \sup B = 1$. However $M = \{-2, 1, 4\}$ and hence $\sup M = 4$ which is not equal to 1.

Note that the counterexample relies on having two negative terms that multiply together to give a large positive term. If we restrict A and B to be subsets of the positive real line, $(0, \infty)$, then the result $\sup M = \sup A \cdot \sup B$ would hold, and could be proved following similar logic to the previous exercise.

4. (a) By dividing through by n, we obtain

$$\left(\frac{3n}{n+3}\right)^2 = \left(\frac{3}{1+\frac{3}{n}}\right)^2$$

and since $\frac{1}{n} \to 0$ as $n \to \infty$, we see that

$$\left(\frac{3n}{n+3}\right)^2 \to \left(\frac{3}{1}\right)^2 = 9.$$

(b) By making use of Example 1 in Section 1, we can write

$$\begin{aligned} \frac{1+2+\ldots+n}{n^2} &= \frac{n(n+1)/2}{n^2} \\ &= \frac{n+1}{2n} \\ &= \frac{1+\frac{1}{n}}{2} \\ &\to 1/2 \end{aligned}$$

as $n \to \infty$.

(c) We first write

$$\frac{a^n - b^n}{a^n + b^n} = \frac{1 - \left(\frac{b}{a}\right)^n}{1 + \left(\frac{b}{a}\right)^n} = \frac{1 - c^n}{1 + c^n}$$

where $c = b/a$. Since $a > b > 0$, we know that $1 > c > 0$. Thus $c^n \to 0$ as $n \to \infty$ by Theorem 9.7(b), and hence $(a^n - b^n)/(a^n + b^n) \to 1$ as $n \to \infty$.

(d) Although 2^n rapidly becomes much bigger that n^2, we must be careful to show this rigorously. One method is to use the binomial theorem to expand for $n \geq 3$ according to

$$\begin{aligned} 2^n &= (1+1)^n \\ &= 1^n + n \cdot 1^{n-1} \cdot 1 + \frac{n(n-1)}{2} \cdot 1^{n-2} \cdot 1^2 + \frac{n(n-1)(n-2)}{6} \cdot 1^{n-3} 1^3 + \ldots \end{aligned}$$

and hence, by neglecting all but one term,

$$2^n > \frac{n(n-1)(n-2)}{6}.$$

Now, for $n \geq 3$, we know that $(n-1) > \frac{n}{2}$ and $(n-2) > \frac{n}{4}$, and hence

$$2^n > \frac{n^3}{24}$$

and therefore $2^n > n^3/24$. Thus for $n \geq 3$, $0 < n^2/2^n < 24/n$, and thus $n^2/2^n \to 0$ as $n \to \infty$ by the Squeezing Lemma.

(e) This can be carried out by introducing a factor that completes the square:

$$\begin{aligned} \sqrt{n+1} - \sqrt{n} &= \left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}}. \end{aligned}$$

Since $\sqrt{n} \to \infty$ as $n \to \infty$, we must have $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to \infty$.

5. (a) Let $s_n = \frac{\sqrt{2}}{n}$ for all $n \in \mathbb{N}$. We know that s_n is irrational, since if $s_n = p/q$ for some integers p and q, then $\sqrt{2} = p/(qn)$, but $\sqrt{2}$ has been shown to be irrational. Now consider an $\epsilon > 0$. We see that

$$|s_n - 0| = \frac{\sqrt{2}}{n}$$

and thus if $n > \sqrt{2}\epsilon$, then $|s_n - 0| < \epsilon$. Hence $s_n \to 0$ as $n \to \infty$.

(b) There are many ways this could be achieved, such as defining s_n as the first digits of π, so that the first few terms would be $3, 3.1, 3.14, 3.141, 3.1415, 3.14159$. However, here a method is presented which shows explicitly how to construct all the numbers in a sequence, and show that they converge to a irrational. Define $s_n = p_n/q_n$ and put $p_1 = 1$ and $q_1 = 1$. Now, define the rest of the sequence recursively by putting

$$p_{n+1} = p_n + 2q_n, \qquad q_{n+1} = p_n + q_n.$$

It is straightforward to see that if $p_n > 0$ and $q_n > 0$, then $p_{n+1} > 0$ and $q_{n+1} > 0$, so by mathematical induction $s_n > 0$ and $q_n \neq 0$ for all n. The first few terms are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}.$$

The last of these is $1.4142156862\ldots$ which differs from $\sqrt{2}$ by 2.12×10^{-6}. Here, we prove that s_n does indeed converge to $\sqrt{2}$. Suppose that s_n differs from $\sqrt{2}$ by an amount Δ_n, so that

$$\frac{p_n}{q_n} - \sqrt{2} = \Delta_n. \tag{1}$$

Then consider how much s_{n+1} differs from $\sqrt{2}$:

$$\begin{aligned}
\Delta_{n+1} &= \frac{p_{n+1}}{q_{n+1}} - \sqrt{2} \\
&= \frac{p_n + 2q_n}{p_n + q_n} - \sqrt{2} \\
&= \frac{\frac{p_n}{q_n} + 2}{\frac{p_n}{q_n} + 1} - \sqrt{2} \\
&= \frac{\Delta_n + \sqrt{2} + 2}{\Delta_n + \sqrt{2} + 1} - \sqrt{2} \\
&= \frac{\Delta_n + \sqrt{2} + 2 - \sqrt{2}(\Delta_n + \sqrt{2} + 1)}{\Delta_n + \sqrt{2} + 1} \\
&= \frac{(1 - \sqrt{2})\Delta_n}{\Delta_n + \sqrt{2} + 1}.
\end{aligned}$$

Since p_n/q_n is positive, we know from Eq. 1 that $\Delta_n + \sqrt{2} > 0$. We also know that $1 < \sqrt{2} < 3/2$ since $1^2 = 1 < 2$ and $(3/2)^2 = 9/4 > 2$. Hence $-1/2 <$

$1-\sqrt{2}<0$. Using these inequalities,

$$\begin{aligned}|\Delta_{n+1}| &= |\Delta_n| \cdot \left|\frac{1-\sqrt{2}}{\Delta_n+\sqrt{2}+1}\right| \\ &\leq |\Delta_n| \cdot \left|\frac{1/2}{1}\right| \\ &\leq \frac{|\Delta_n|}{2}.\end{aligned}$$

Hence by mathematical induction, $|\Delta_n| \leq |\Delta_1|(1/2)^{n-1}$, and thus by Theorem 9.7(b), $|\Delta_n| \to 0$ as $n \to \infty$. Hence Eq. 1 shows that $s_n = p_n/q_n \to \sqrt{2}$ as $n \to \infty$.

6. We can rewrite a term in the sequence as a product of fractions,

$$s_n = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right).$$

Each of these fractions is less than or equal to one, and the first is equal to $\frac{1}{n}$. Thus $s_n \leq \frac{1}{n}$. Now choose $\epsilon > 0$. We see that

$$|s_n - 0| = s_n \leq \frac{1}{n}$$

and thus we see that for all $n > \epsilon^{-1}$, $|s_n - 0| < \epsilon$. Hence $\lim_{n\to\infty} s_n = 0$.

7. An arbitrary polynomial can be written as a sum

$$p(x) = \sum_{j=0}^{k} a_j x^j$$

where $a_j \in \mathbb{R}$ and $a_k \neq 0$. To begin, we show by induction that if $s_n \to s$ as $n \to 0$, then $(s_n)^j \to s^j$ for all $j \in \mathbb{N} \cup \{0\}$. Consider the base case when $j = 0$. Since $(s_n)^0 = 1$ for all n, this is a constant sequence, and thus converges to 1, which is equal to s^0.

Now assume the result is true for j and consider the case for $j+1$. We can define $(s_n)^{j+1} = (s_n)^j \cdot s_n$ and thus by Theorem 9.4. we know that $(s_n)^j \cdot s_n \to s^j \cdot s = s^{j+1}$ as $n \to \infty$. Hence the induction step holds, and by mathematical induction $(s_n)^j \to s^j$ for all $j \in \mathbb{N} \cup \{0\}$.

Now, if a_j is a constant, then we know that $a_j(s_n)^j \to a_j s^j$ by Theorem 9.2. Finally, by applying Theorem 9.3, we see that $p(s_n) \to p(s)$ as $n \to \infty$.

8. We begin by proving that $s_n = f(n)$ which is defined according to

$$f(n) = 2 - (2-t)2^{1-n}.$$

Consider the case when $n = 1$:

$$f(1) = 2 - (2-t)2^{1-1} = 2 - (2-t) = t$$

and thus $s_1 = f(1)$. Now assume that the result is true for n and consider the case for $n+1$:

$$\begin{aligned} s_{n+1} &= 1 + \frac{s_n}{2} \\ &= 1 + 1 - \frac{(2-t)2^{1-n}}{2} \\ &= 2 - (2-t)2^{1-(n+1)}. \end{aligned}$$

and thus $s_{n+1} = f(n+1)$. Hence by mathematical induction, $s_n = f(n)$ for all $n \in \mathbb{N}$.

Now, by Theorem 9.7(b), we know that $a^n \to 0$ if $|a| < 1$. Hence, by using Theorems 9.2 and 9.3 about the scaling and addition of sequences we know that $s_n \to 2 - (2-t) \cdot 0 = 2$.