Math 104: Homework 11 solutions

1. (a) If $x \leq 0$, then

$$F(x) = \int_0^x 0\,dt = 0.$$

If $0 < x \leq 1$, then

$$F(x) = \int_0^x t\,dt = \frac{x^2}{2}.$$

If $x > 1$, then

$$F(x) = \int_0^x t\,dt = \int_0^1 tdt + \int_1^x 4dt = \frac{1}{2} + 4(x-1).$$

(b) The function is plotted in Fig. 1. The function is continuous, which follows from the second Fundamental Theorem of Calculus.

(c) At $x = 0$,

$$\lim_{a\to 0^+} \frac{F(0) - F(a)}{0-a} = \lim_{a\to 0^+} \frac{a^2}{a} = 0$$

and

$$\lim_{a\to 0^-} \frac{F(0) - F(a)}{0-a} = \lim_{a\to 0^-} \frac{0}{0-a} = 0.$$

Since the positive and negative limits agree, the function is differentiable at 0. At $x = 1$,

$$\lim_{a\to 1^+} \frac{F(1) - F(a)}{1-a} = \lim_{a\to 1^+} \frac{4(a-1)}{a-1} = 4$$

and

$$\begin{aligned}\lim_{a\to 1^-} \frac{F(1) - F(a)}{1-a} &= \lim_{a\to 1^-} \frac{\frac{1}{2} - \frac{a^2}{2}}{1-a} \\ &= \lim_{a\to 1^-} \frac{(1-a)(1+a)}{2(1-a)} = 1.\end{aligned}$$

Since the two limits do not agree, F is not differentiable at this point. Thus F' is defined on $\mathbb{R}/\{1\}$ and

$$F'(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ x & \text{for } 0 < x < 1, \\ 4 & \text{for } x > 1. \end{cases}$$

These results are consistent with the second Fundamental Theorem of Calculus, which states that if f is continuous at a point x_0, then F is differentiable at that point and $F'(x_0) = f(x_0)$.

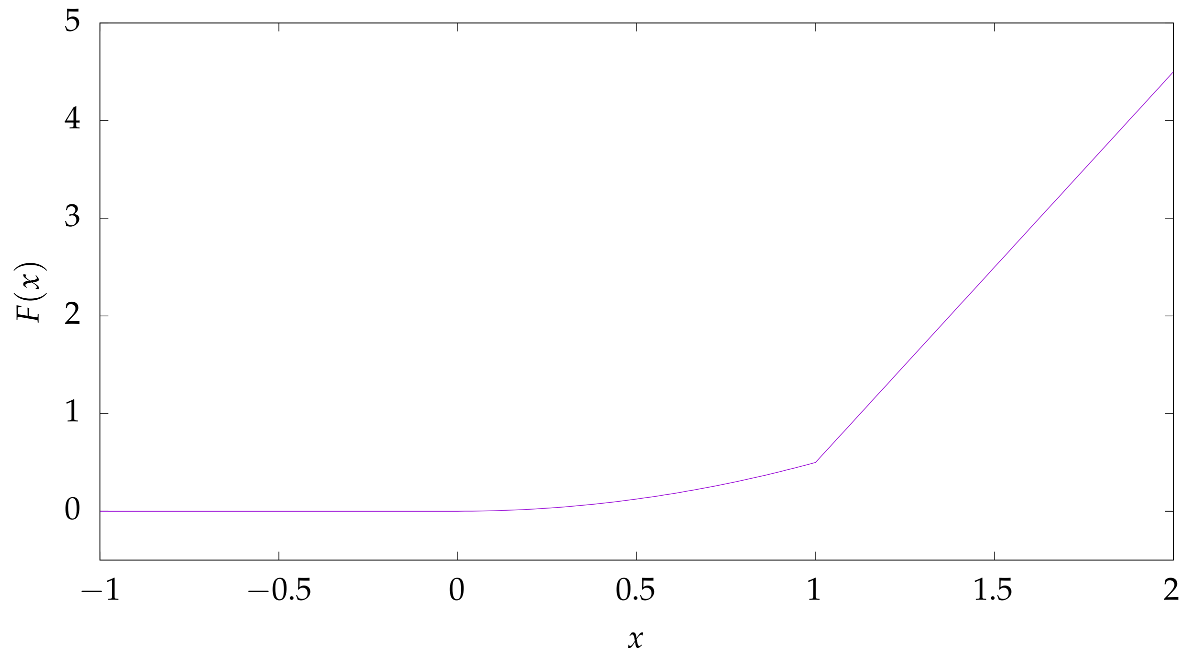

Figure 1: A graph of the function $F(x) = \int_0^x f(t)dt$ used in question 1.

2. Let $f(x)$ be a continuous real-valued function on $[a,b]$. Suppose that $\int_a^b f(x)g(x)dx = 0$ for all continuous functions g, but that $f(x)$ is not constant. Then there exists an $x_0 \in [a,b]$ such that $f(x_0) \neq 0$. Assume that $f(x_0) > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}$$

and hence that

$$f(x) > \frac{f(x_0)}{2}.$$

Thus there exists an open interval $(c,d) \subseteq [a,b]$ in which f is greater than $f(x_0)$. Consider

$$g(x) = \min\left\{\frac{d-c}{2} - \left|x - \frac{c+d}{2}\right|, 0\right\}.$$

Then if $x < c$,

$$\begin{aligned} g(x) &= \min\left\{\frac{d-c}{2} + x - \frac{c+d}{2}, 0\right\} \\ &= \min\{x - c, 0\} = 0. \end{aligned}$$

Similarly, $g(x) = 0$ if $x > d$. Now

$$\begin{aligned} \int_a^b g(x)dx &= \int_c^{(c+d)/2} \left(\frac{d-c}{2} + x - \frac{c+d}{2}\right) dx + \int_{(c+d)/2}^d \left(\frac{d-c}{2} - x + \frac{c+d}{2}\right) dx \\ &= \int_c^{(c+d)/2} (x-c)dx + \int_{(c+d)/2}^d (d-x)dx \\ &= \frac{(d-c)^2}{4} > 0 \end{aligned}$$

However, $f(x)g(x) \geq f(x_0)g(x)/2$ for all $x \in [a,b]$, and hence

$$\int_a^b f(x)g(x)dx \geq \frac{f(x_0)}{2}\int_a^b g(x)dx = \frac{f(x_0)}{2}\frac{(d-c)^2}{4} > 0.$$

If $f(x_0) < 0$, then the same argument can be applied to $-f$ to show that $\int_a^b fg < 0$. Hence there exists a g such that $\int_a^b fg \neq 0$, which is a contradiction. Thus $f(x) = 0$ for all $x \in [a,b]$.

3. (a) Choose $\alpha \in (a,b)$. Then

$$\begin{aligned} \int_a^b f &= \int_a^\alpha f + \int_\alpha^b f \\ &= \lim_{c \to a^+} \int_c^\alpha f + \lim_{d \to b^-} \int_\alpha^d f \end{aligned} \tag{1}$$

and similarly

$$\int_a^b g = \lim_{c\to a^+} \int_c^\alpha g + \lim_{d\to b^-} \int_\alpha^d g. \tag{2}$$

If $\int_a^b g < \infty$, then both of the terms in the above expression must be finite. Consider the two terms integrated over the range from c to α. Since $0 \le f(x) \le g(x)$, then for all c

$$\int_c^\alpha f \le \int_c^\alpha g \le \int_a^\alpha g.$$

Thus the integral of f is bounded above, and hence

$$\lim_{c\to a^+} \int_c^\alpha f \le \int_a^\alpha g < \infty.$$

The same logic can be applied to the term from α to d. Hence $\int_a^b f < \infty$.

(b) If $\int_a^b f = \infty$, then at least one of the terms in Eq. 1 is equal to ∞; suppose it is the first. Then for all $M > 0$ there exists E such that $a < c < E$ implies

$$\int_c^\alpha f > M.$$

Hence

$$\int_c^\alpha g \ge \int_c^\alpha f > M,$$

and thus $\lim_{c\to a^+} g = \infty$. Hence from Eq. 2, $\int_a^b g = \infty$. The same logic could be applied if the second term in Eq. 1 is infinite, and thus the result is true for all cases.

4. Consider the sequence of functions

$$f_n(x) = \begin{cases} \frac{1}{2n} & \text{if } |x| \le n \\ 0 & \text{if } |x| > n. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} f_n(x) = \lim_{c\to\infty} \int_0^c f_n(x)dx + \lim_{d\to-\infty} \int_d^0 f_n(x)dx.$$

If $c > n$ and $d < -n$, then

$$\int_0^c f_n(x)dx = n \cdot \frac{1}{2n} = \frac{1}{2}, \qquad \int_{-d}^0 f_n(x)dx = n \cdot \frac{1}{2n} = \frac{1}{2}.$$

and thus

$$\int_{-\infty}^{\infty} = \frac{1}{2} + \frac{1}{2} = 1.$$

To show that $f_n \to 0$ uniformly, consider any $\epsilon > 0$. Then there exists and N such that $\frac{1}{2N} < \epsilon$. For $n > N$, and $x \in \mathbb{R}$

$$|f_n(x) - 0| \leq \left| \frac{1}{2n} - 0 \right| = \frac{1}{2n} < \epsilon.$$

and thus $f_n \to 0$ uniformly.

5. (a) The tangent can be defined for $(-\frac{\pi}{2}, \frac{\pi}{2})$ in terms of sine and cosine as

$$\tan x = \frac{\sin x}{\cos x}.$$

Since $\cos x > 0$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and sine and cosine are differentiable, the quotient theorem can be applied to show that

$$\begin{aligned} \tan' x &= \frac{\cos x \sin' x - \sin x \cos' x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x}. \end{aligned}$$

Hence $\tan' x > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, so $\tan x$ is strictly increasing. As $x \to \pi/2$, $\cos x \to 0$ and $\sin x \to \pi/2$. Consider any $M > 0$. Then there exists a $\delta_1 > 0$ such that $\frac{\pi}{2} - \delta_1 < x < \frac{\pi}{2}$ implies that

$$0 < \cos x < \frac{1}{2M}.$$

Similarly, there exists a $\delta_2 > 0$ such that for $\frac{\pi}{2} - \delta_2 < x < \frac{\pi}{2}$,

$$0 < \sin x < \frac{1}{2}.$$

Hence if $\delta = \min\{\delta_1, \delta_2\}$, then $\frac{\pi}{2} - \delta < x < \frac{\pi}{2}$ implies that $\tan x > \frac{2M}{2} = M$. Hence $\lim_{x \to \pi/2} \tan x = \infty$. Since sine is an odd function and cosine is an even function, then the tangent must be an odd function, so $\lim_{x \to -\pi/2} \tan x = -\infty$. Hence $\tan x$ is not bounded above or below.

(b) By the previous results, tan is a one-to-one differentiable function from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. Hence by the inverse function theorem, there exists a function $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ which is differentiable and satisfies

$$\tan^{-1}(\tan y) = y.$$

By applying the chain rule,

$$(\tan^{-1})'(\tan y) \tan' y = 1$$

and thus if $x = \tan y$,

$$(\tan^{-1})'(x) = \cos^2 y.$$

By making use of the trigonometric identity $\cos^2 y + \sin^2 y = 1$,

$$(\tan^{-1})'(x) = \frac{\cos^2 y}{\cos^2 y + \sin^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1+x^2}.$$

(c) Let

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

The radius of convergence is $R = 1$, and thus the series converges for $|x| < 1$. Since power series can be differentiated,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}.$$

Thus, over the range $(-1, 1)$, $f'(x) = (\tan^{-1})'(x)$, and by Corollary 29.5 $f(x) = \tan^{-1}(x) + C$ for some $C \in \mathbb{R}$. Since $f(0) = \tan^{-1}(0) = 0$, it follows that $f(x) = \tan^{-1}(x)$ for $x \in (-1, 1)$.

(d) By the inverse function theorem $\tan^{-1}$ is continuous on the range $(-\frac{\pi}{2}, \frac{\pi}{2})$. At $x = 1$,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

which converges by the alternating series theorem. Abel's theorem states that if a power series f converges at $x = R$, then f is continuous at $x = R$. Since $f(x) = \tan^{-1}(x)$ for $|x| < R$, it follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = f(1) = \lim_{x \to 1} f(x) = \lim_{x \to 1} \tan^{-1} x = \tan^{-1}(1) = \frac{\pi}{4}.$$

(e) By direct calculation $(5 + i)^4(239 - i) = 114244 + 114244i$. When complex numbers are multiplied, their arguments are additive, and hence

$$4 \arg(5 + i) + \arg(239 - i) = \arg(114244 + 114244i).$$

so

$$4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}.$$

6. (a) I_0 is given by

$$I_0 = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}$$

and I_1 is given by

$$I_1 = \int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} = 0 - (-1) = 1.$$

(b) By using integration by parts,

$$\begin{aligned}
I_{n+2} &= \int_0^{\pi/2} \sin^{n+2} x\, dx \\
&= \int_0^{\pi/2} \sin x(\sin^{n+1} x)\, dx \\
&= \Big[\cos x \sin^{n+1} x\Big]_0^{\pi/2} + \int_0^{\pi/2} (n+1)\cos x(\sin^n x \cos x)\, dx \\
&= 0 + (n+1)\int_0^{\pi/2} \sin^n x(1-\sin^2 x)\, dx \\
&= (n+1)(I_n + I_{n+2})
\end{aligned}$$

from which it follows that $(n+2)I_{n+2} = (n+1)I_n$.

(c) Since $\sin^{2m+1} x \leq \sin^{2m} x$ for all $x \in [0, \pi/2]$, it follows that $I_{2m+1} \leq I_{2m}$. Similarly, by using the previously derived identity,

$$I_{2m} \leq I_{2m-1} = I_{2m+1}\frac{2m+1}{2m} = \left(1 + \frac{1}{2m}\right) I_{2m+1}.$$

(d) By repeated application of the identity $(n+2)I_{n+2} = (n+1)I_n$,

$$\begin{aligned}
\frac{\pi}{2} &= I_0 \\
&= \frac{2}{1} I_2 \\
&= \frac{2}{1}\frac{4}{3} I_4 \\
&= \frac{2}{1}\frac{4}{3}\frac{6}{5}\cdots\frac{2m}{2m-1} I_{2m} \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
1 &= I_1 \\
&= \frac{3}{2} I_3 \\
&= \frac{3}{2}\frac{5}{4} I_5 \\
&= \frac{3}{2}\frac{5}{4}\frac{7}{6}\cdots\frac{2m+1}{2m} I_{2m+1}. \qquad (4)
\end{aligned}$$

By using the inequalities in the previous section,

$$1 \leq \frac{I_{2m}}{I_{2m+1}} \leq 1 + \frac{1}{2m}.$$

Since $\frac{1}{2m} \to 1$, the squeezing lemma can be applied to show that

$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

By dividing Eq. 3 by Eq. 4, it can be shown that

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}\frac{I_{2m}}{I_{2m+1}}$$

and hence

$$\frac{\pi}{2\frac{I_{2m}}{I_{2m+1}}} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}.$$

Now take limits on both sides. By sequence limit theorems, the left hand side must converge to $\pi/2$, and hence the right hand side must converge to the same limit also. Hence

$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}.$$