Math 104: Homework 10 solutions

1. (a) The first derivative is

$$f'(x) = \frac{1}{x+1}$$

and the nth derivative is given by

$$f^{(n)} = \frac{(n-1)!(-1)^{n-1}}{(x+1)^n}.$$

Hence the Taylor series expansion at $x = 0$ is given by

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{n!} + R_n(x) \\ &= \sum_{k=1}^{n-1} \frac{(n-1)!}{n!}(-1)^{n-1} + R_n(x) \\ &= \sum_{k=1}^{n-1} \frac{(-1)^{n-1}}{n} + R_n(x). \end{aligned}$$

(b) By using Taylor's Theorem, the remainder is given by

$$R_n(x) = \frac{f^{(n)}(y)}{n!}x^n = \frac{(-1)^{n-1}}{n(y+1)^n}x^n = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+y}\right)^n$$

where y is between 0 and x. If $0 < x < 1$, then $y > 0$, so $0 < x/(1+y) < x$ and hence $R_n(x) \to 0$ as $n \to 0$. If $-1/2 < x < 0$, then $1 + y > 1/2$, so $x/(1+y) < (1/2)2 = 1$ and hence $R_n(x) \to 0$ as $n \to 0$. Hence the Taylor series agrees with f in the range $-1/2 < x < 1$.

2. Suppose that there exists a $y \in [a, b]$ such that $f(y) > 0$. Suppose $y = a$; then since f is continuous, there exists a $\delta > 0$ such that $|x - a| < \delta$ implies that $|f(a) - f(x)| < f(a)/2$, in which case $f(a + \delta/2) > 0$. Similarly, if $y = b$, then there exists a $\delta > 0$ such that $f(b - \delta/2) > 0$. Hence there must exist an $x_0 \in (a, b)$ such that $f(x_0) > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|x - x_0| < \delta$ implies that

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}$$

and hence

$$f(x) > \frac{f(x_0)}{2}. \tag{1}$$

Since $x_0 \in (a, b)$ it is always possible to find a $\delta > 0$ to that $(x_0 - \delta, x_0 + \delta) \subset [a, b]$. Consider the partition $P = \{a = t_0 < t_1 < t_2 < t_3 = b\}$ where $t_1 = x_0 - \frac{\delta}{2}$ and $t_2 = x_0 - \frac{\delta}{2}$. Then

$$L(f, P) = \sum_{k=1}^{3} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k])$$

Since $f(x) \geq 0$ for all $x \in [a, b]$, then $m(f, [a, t_1]) \geq 0$ and $m(f, [t_2, b]) \geq 0$. In addition, by reference to Eq. 1, $m(f, [t_1, t_2]) \geq f(x_0)/2$, and hence

$$L(f, P) \geq 0 + \left(x_0 + \frac{\delta}{2} - x_0 + \frac{\delta}{2}\right) \frac{f(x_0)}{2} + 0 = \frac{\delta x_0}{2} > 0.$$

Hence $\int_a^b f > 0$, which is a contradiction. Hence, $f(x) = 0$ for all $x \in [a, b]$.

3. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

defined on $[0, 1]$. Then $f(x)^2 = 1$ for all $x \in [0, 1]$, and this function is integrable.

Consider any partition $P = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$. Then $m(f, [t_{k-1}, t_k]) = -1$ for $k = 1, \ldots, n$ since any interval of finite length must contain an irrational number. Similarly, $M(f, [t_{k-1}, t_k]) = 1$ for $k = 1, \ldots, n$ since any interval of finite length must also contain a rational number. Thus

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = -\sum_{k=1}^{n} (t_k - t_{k-1}) = -(1 - 0) = -1$$

and

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k - t_{k-1}) = 1 - 0 = 1.$$

Since this is true for any partition, it follows that $U(f) = 1$, and $L(f) = -1$, so f is not integrable.

4. (a) Let f be a bounded function on $[a, b]$, so that there exists a $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$. Recall the result from Ross Exercise 4.14, that for two sets A and B, then the set C of sums $a + b$ where $a \in A$ and $b \in B$ satisfies

$$\sup C = \sup A + \sup B.$$

For any interval $I \subseteq [a, b]$,

$$\begin{aligned} M(f^2, I) - m(f^2, I) &= \sup\{f(x)^2 : x \in I\} - \inf\{f(x)^2 : x \in I\} \\ &= \sup\{f(x)^2 : x \in I\} + \sup\{-f(x)^2 : x \in I\} \\ &= \sup\{f(x)^2 - f(y)^2 : x \in I, y \in I\} \\ &= \sup\{(f(x) - f(y))(f(x) + f(y)) : x \in I, y \in I\} \\ &\leq 2B \sup\{f(x) - f(y) : x \in I, y \in I\} \\ &= 2B(\sup\{f(x) : x \in I\} - \inf\{f(x) : x \in I\}). \end{aligned}$$

Now consider any partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. Then

$$\begin{aligned} U(f^2, P) - L(f^2, P) &= \sum_{k=1}^{n} (t_k - t_{k-1})(M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k])) \\ &\leq 2B \sum_{k=1}^{n} (t_k - t_{k-1})(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) \\ &= 2B[U(f, P) - L(f, P)]. \end{aligned}$$

(b) If f is integrable, then for all $\epsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \frac{\epsilon}{2B}.$$

Thus by using the above inequality,

$$U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) < \epsilon$$

and hence f^2 is integrable.

5. (a) Since f and g is integrable, then $f + g$ and $f - g$ are integrable by Theorem 33.3. The result from the previous question shows that $(f + g)^2$ and $(f - g)^2$ are integrable also. Applying Theorem 33.3 again shows that

$$fg = \frac{(f + g)^2 - (f - g)^2}{4}$$

is integrable.

(b) By Theorem 33.5, $|f - g|$ is integrable, and thus by Theorem 33.3,

$$\max(f, g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$$

is integrable. Since $-f$, and $-g$ are integrable,

$$\min(f, g) = -\max(-f, -g)$$

is integrable also.

6. (a) For any two numbers $u, v \in \mathbb{R}$,

$$(u + v)^2 \geq 0$$

so

$$u^2 + 2uv + v^2 \geq 0$$

and hence $uv \leq (u^2 + v^2)/2$. Consider two integrable functions f and g on $[a, b]$, where $\int_a^b f^2 = \int_a^b g^2 = 1$. Then by Exercise 33.8, fg is integrable. Since $f(x)g(x) \leq (f(x)^2 + g(x)^2)/2$, Theorem 33.4 shows that

$$\int_a^b fg \leq \int_a^b \frac{f^2 + g^2}{2}$$

and hence

$$\int_a^b fg \leq \left(\int_a^b \frac{f^2}{2}\right) + \left(\int_a^b \frac{g^2}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1.$$

(b) Consider two integrable functions f and g on $[a, b]$. Define $C = \int_a^b f^2$ and $D = \int_a^b g^2$. Suppose initially that both $C > 0$ and $D > 0$, then define $F(x) = f(x)/\sqrt{C}$ and $G(x) = g(x)/\sqrt{D}$ for $x \in [a, b]$. Hence

$$\int_a^b F^2 = \int_a^b \frac{f^2}{C} = \frac{1}{C} \int_a^b f^2 = 1$$

and similarly $\int_a^b G^2 = 1$, so the inequality of the previous section can be applied to show that

$$\int_a^b FG \leq 1.$$

Thus

$$\int_a^b \frac{fg}{CD} \leq 1$$

so

$$\int_a^b fg \leq CD = \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$

Since the same inequality would hold if applied to $-f$ and g, it follows that

$$\left|\int_a^b fg\right| \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$

Now consider the case when $C = 0$ or $D = 0$. If f and g are continuous, then the result follows quickly from question 3. If $C = 0$, then $f(x)^2 = 0$ for all $x \in [a, b]$, and hence $f(x) = 0$ for all $x \in [a, b]$, in which case the Schwarz inequality is satisfied. Similarly, if $D = 0$, then $g(x) = 0$ for all $x \in [a, b]$, and the Schwarz inequality is satisfied.

However, if f and g are not assumed to be continuous, the result requires a more direct approach. Suppose $C = 0$ and $D \neq 0$, and consider any $\epsilon > 0$. Then there exists a partition P such that

$$U(f^2, P) - L(f^2, P) < \epsilon.$$

It is known that $L(f^2, P) \geq 0$ since $f(x)^2 \geq 0$ for all x. Since $L(f^2, P) \leq \int_a^b f^2 = 0$, it follows that $L(f^2, P) = 0$. If $U(f^2, P) = 0$, then that implies that $f(x) = 0$ for all $x \in [a, b]$, and hence the Schwarz inequality is satisfied. Otherwise, consider the step function

$$h_P(x) = \begin{cases} M(|f|, [t_{k-1}, t_k]) & \text{for } x \in [t_{k-1}, t_k), \\ M(|f|, [t_{n-1}, b]) & \text{for } x = b. \end{cases}$$

By construction,

$$\int_a^b h_P^2 = U(f^2, P).$$

and since $U(f^2, P) > 0$, the Schwarz inequality can be applied to h_P and g, to show that

$$\begin{aligned} \left| \int_a^b fg \right| &\leq \int_a^b |fg| \\ &\leq \int_a^b |h_P g| \\ &\leq \left(\int_a^b h_P^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2} \\ &= \left(U(f^2, P) \right)^{1/2} D \\ &< D\sqrt{\epsilon}. \end{aligned}$$

Since this is true for arbitrary $\epsilon > 0$, it follows that $|\int_a^b fg| = 0$, and thus the Schwarz inequality is satisfied. The same argument can then be repeated to show that the result also holds for $D = 0$. Hence the Schwarz inequality is satisfied for all functions.

(c) Consider the three properties of a metric:

M1. Consider $f \in X$. Then

$$d(f, f) = \left(\int_a^b |f - f|^2 \right)^{1/2} = \left(\int_a^b 0 \right)^{1/2} = 0.$$

Suppose $f, g \in X$ and $d(f, g) = 0$. Then

$$0 = \int_a^b |f - g|^2$$

and by the result from question 3, since $|f - g|^2$ is continuous, it follows that $|f(x) - g(x)|^2 = 0$ for all $x \in [a, b]$. Hence $f(x) = g(x)$ for all $x \in [a, b]$. Thus $d(f, g) = 0$ if and only if $f = g$.

M2. For any two $f, g \in X$,

$$d(f,g) = \left(\int_a^b |f-g|^2\right)^{1/2} = \left(\int_a^b |g-f|^2\right)^{1/2} = d(g,f)$$

so d is symmetric.

M3. Consider $f, g, h \in X$. If $d(f,h) = 0$, then the inequality $d(f,h) \leq d(f,g) + d(g,h)$ is immediately satisfied. Otherwise consider

$$\begin{aligned} d(f,h)(d(f,g)+d(g,h)) &= \left(\int_a^b |f-g|^2\right)^{1/2}\left(\int_a^b |f-h|^2\right)^{1/2} \\ &\quad + \left(\int_a^b |g-h|^2\right)^{1/2}\left(\int_a^b |f-h|^2\right)^{1/2} \\ &\geq \left|\int_a^b |f-g|\cdot|f-h|\right| + \left|\int_a^b |g-h|\cdot|f-h|\right| \\ &= \int_a^b |f-g|\cdot|f-h| + \int_a^b |g-h|\cdot|f-h| \\ &= \int_a^b (|f-g|+|g-h|)\cdot|f-h|. \end{aligned}$$

The usual triangle inequality can be applied to show that

$$d(f,h)(d(f,g)+d(g,h)) \geq \int_a^b |f-h|\cdot|f-h| = d(f,h)^2$$

and hence, since $d(f,h) > 0$,

$$d(f,g) + d(g,h) \geq d(f,h)$$

so the triangle inequality is satisfied.

Hence d is a metric.