Math 104: Solutions to sample final problems

1. For the first series

$$\beta = \limsup |a_n|^{1/n} = \limsup |n|^{3/n} = 1$$

so the radius of convergence is $R = {}^{1}/_{\beta} = 1$. At $n = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

which converges (and can be verified using the integral test). At $n = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

which converges by the alternating series test. Hence the exact interval of convergence is $[-1, 1]$. For the second series, only every third term is non-zero, and thus

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_{3n}|^{1/3n} = \lim_{n\to\infty} \left| \frac{1}{2n} \right|^{1/3n} = 1$$

so the radius of convergence is 1. At $x = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

which diverges. At $x = -1$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}$$

which converges by the alternating series test. Hence the exact interval of convergence is $(-1, 1]$. For the third series

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_{2n!}|^{1/2n!} = \lim_{n\to\infty} 1 = 1$$

and hence the radius of convergence is 1. At $x = 1$

$$\sum_{n=0}^{\infty} x^{2n!} = \sum_{n=0}^{\infty} 1$$

which diverges. Since the series is even, it diverges at $x = -1$ also. Hence the exact interval of convergence is $(-1, 1)$.

2. (a) Suppose that $s_n \to s$. Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|s_n - s| < \epsilon.$$

To show that $s_n^2 \to s^2$, consider any $\epsilon > 0$. First, note that

$$|s_n^2 - s^2| = |s_n - s| \cdot |s_n + s|.$$

There exists an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies that $|s_n - s| < 1$, and hence that

$$|s_n + s| \leq |s_n| + |s| < (|s| + 1) + |s| = 2|s| + 1.$$

Similarly, there exists an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that

$$|s_n - s| < \frac{\epsilon}{2|s| + 1}.$$

Hence, if $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$\begin{aligned} |s_n^2 - s^2| &\leq |s_n - s| \cdot |s_n + s| \\ &< \frac{\epsilon}{2|s| + 1}(2|s| + 1) = \epsilon. \end{aligned}$$

(b) A function f is continuous at a point x if for all sequences (x_n) such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$. By part (a), for all $x \in \mathbb{R}$, if $x_n \to x$, then $x_n^2 \to x^2$. Thus f is continuous for all $x \in \mathbb{R}$.

Alternatively, to use the ϵ-δ property, choose $\epsilon > 0$, and fix $x_0 \in \mathbb{R}$. Then

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|.$$

If $|x - x_0| < 1$, then $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$. Pick $\delta = \min\{\frac{\epsilon}{2|x_0|+1}, 1\}$. Then $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0)| < \frac{\epsilon}{2|x_0| + 1}(2|x_0| + 1) = \epsilon$$

and hence f is continuous at x_0. Since x_0 is arbitrary, f is continuous on $\mathbb{R}$.

3. By the Mean Value Theorem, there exists a $c \in (r, s)$ such that

$$f'(c) = \frac{f(s) - f(r)}{s - r} = 0$$

and there exists and $d \in (s, t)$ such that

$$f'(d) = \frac{f(t) - f(s)}{t - s} = 0.$$

By construction, $d > c$. Hence there exists an $x \in (c, d) \subseteq (0, 1)$ such that

$$f''(x) = \frac{f'(d) - f'(c)}{d - c} = 0.$$

4. (a) If $\sum_{n=0}^{\infty} a_n$ converges to a limit a, then for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $k > N$ implies

$$\left| a - \sum_{n=0}^{k} a_n \right| < \epsilon.$$

Now consider any $\epsilon > 0$ and define N as above. If $k > N/2$, then

$$\left| a - \sum_{n=0}^{k} b_n \right| = \left| a - \sum_{n=0}^{2k+1} a_n \right| < \epsilon.$$

Hence $\sum_{n=0}^{\infty} b_n$ converges.

(b) Suppose $a_n = (-1)^n$. Then

$$\sum_{n=0}^{N} a_n = \begin{cases} 1 & \text{if } N \text{ is even,} \\ 0 & \text{if } N \text{ is odd;} \end{cases}$$

this series diverges. However $b_n = 0$ for all n, and thus $\sum_{n=0}^{\infty} b_n$ converges.

5. If $f(a)f(b) < 0$, then it follows that either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. In either case, zero lies between $f(a)$ and $f(b)$, and thus the intermediate value theorem can be applied to show that there exists an $x \in (a, b)$ such that $f(x) = 0$.

6. Let f be a real-valued function defined on an interval $[0, b]$ as

$$f(x) = \begin{cases} x & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

Consider a partition $P = \{0 = t_0 < t_1 < \ldots < t_n = b\}$. First note that

$$m(f, [t_{k-1}, t_k]) = 0, \qquad M(f, [t_{k-1}, t_k]) = t_k$$

for all $k = 1, \ldots, n$. The first expression follows because any interval must contain an irrational number, whereas the second expression follows because the interval must take rational values arbitrarily close to t_k. Hence

$$U(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = \sum_{k=1}^{n} (t_k - t_{k-1}) t_k.$$

For a general partition this cannot be calculated explicitly, but it will always be strictly positive, since each term in the sum is strictly positive. The lower Darboux sum is given by

$$L(f, P) = \sum_{k=1}^{n} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = 0.$$

To show that f is not integrable on $[0, b]$, consider any partition P. Then

$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n}(t_k - t_{k-1})t_k \\
&\geq \sum_{k=1}^{n}(t_k - t_{k-1})\frac{(t_k + t_{k-1})}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n}(t_k^2 - t_{k-1}^2) \\
&= \frac{t_n^2 - t_0^2}{2} = \frac{b^2}{2}.
\end{aligned}$$

Thus for any partition,

$$U(f,P) - L(f,P) \geq \frac{b^2}{2}$$

and thus f is not integrable on $[0, b]$.

7. Suppose that $\sum_{n=1}^{\infty} f(n)$ converges, and consider the integral

$$\int_1^b f(x)\,dx.$$

Let k be the largest integer such that $k < b$. Hence $b - k \leq 1$. Consider the partition $P = \{1 = t_0 < t_1 < \ldots < t_{k-1} < t_k = b\}$, where $t_i = i + 1$ for $i = 0, \ldots, k-1$. Then

$$\begin{aligned}
U(f,P) &= \sum_{j=1}^{k} M(f,[t_{j-1},t_j])(t_j - t_{j-1}) \\
&= \sum_{j=1}^{k} f(t_{j-1})(t_j - t_{j-1}) \\
&= f(k)(b-k) + \sum_{j=1}^{k-1} f(j) \\
&\leq f(k) + \sum_{j=1}^{k-1} f(j) \\
&= \sum_{j=1}^{k} f(j)
\end{aligned}$$

so

$$\int_1^b f \leq U(f,P) \leq \sum_{j=1}^{k} f(j) \leq \sum_{j=1}^{\infty} f(j).$$

Since the integral is an increasing function and is bounded above, it follows that it must converge.

Now suppose that $\int_1^x f$ converges, and consider $\sum_{n=1}^N f(n)$. Consider the partition $P = \{1 = t_0 < t_1 < \ldots < t_N - 1 = N\}$ where $t_i = i + 1$ for all $i = 0, \ldots, N-1$. Then

$$\begin{aligned} L(f,P) &= \sum_{j=1}^{N-1} m(f,[t_{j-1},t_j])(t_j - t_{j-1}) \\ &= \sum_{j=1}^{N-1} m(f,[j,j+1]) \\ &= \sum_{j=1}^{N-1} f(j+1) \\ &= \sum_{j=2}^{N} f(j) \end{aligned}$$

and hence

$$\sum_{j=1}^{N} f(j) = f(1) + L(f,P) \leq f(1) + \int_1^N f \leq f(1) + \int_1^\infty f.$$

Since $\sum_{j=1}^N f(j)$ is increasing and bounded above, then it converges.

8. (a) From the definition, it is clear that $d(x,y) = 0$ if and only if $x = y$, and that $d(x,y) = d(y,x)$. To prove the triangle inequality, consider any $x, y, z \in \mathbb{R}$. If $x = z$, then $d(x,z) = 0$, and since $d(x,y) + d(y,z) \geq 0$, the triangle inequality is satisfied. If $x \neq z$, then $d(x,z) = 1$. Either $y \neq z$ or $y \neq x$, and hence $d(x,y) + d(y,z) \geq 1$, so the triangle inequality is satisfied. Hence d is a metric.

   (b) The neighborhood of radius $1/2$ at 0 is

$$\{x \in \mathbb{R} : d(0,x) < 1/2\} = \{0\}.$$

   It only contains zero, since all other points are a distance of 1 away.

   (c) Consider an arbitrary set $S \subseteq \mathbb{R}$. To show that S is open, consider any $x \in S$. Then, by the same argument as in (b), $N_{1/2}(x) = \{x\} \subseteq S$ so x is an interior point. Since all points are interior, it follows that S is open.

   Consider an open cover $\mathcal{S}$ of S. Suppose S has finitely many points, so that $S = \{s_1, s_2, \ldots, s_n\}$. Then there exist sets $S_1, \ldots, S_n \in \mathcal{S}$ such that $s_k \in S_k$ for $k = 1, \ldots, n$, and it follows that $\{S_k : k = 1, \ldots n\}$ is a finite subcover. Hence S is compact.

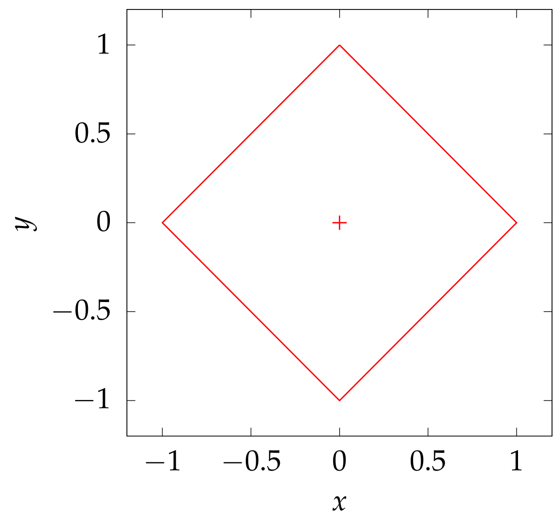

Figure 1: A plot of the neighborhood of radius 1 at $(0,0)$, considered in Problem 9(b).

Suppose S has infinitely many points. Consider the cover $\mathcal{S} = \{\{x\} : x \in S\}$. By above, the sets $\{x\}$ are open. Consider a subcover $\mathcal{T} \subseteq \mathcal{S}$. For any $x \in S$, $\{x\}$ must be $\mathcal{T}$, since x is not an element of any other set in $\mathcal{S}$. Thus $\mathcal{T} = \mathcal{S}$. Hence $\mathcal{S}$ has no finite subcover and $\mathcal{S}$ is not compact.

9. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ be in $\mathbb{R}^2$. Consider the function

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

(a) Consider the three properties of a metric:

M1. Note that

$$d(\mathbf{x}, \mathbf{x}) = |x_1 - x_1| + |x_2 - x_2| = 0$$

and if $d(\mathbf{x}, \mathbf{y}) = 0$, then

$$0 = |x_1 - y_1| + |x_2 - y_2|$$

from which it follows that $x_1 = y_1$ and $x_2 = y_2$, so $\mathbf{x} = \mathbf{y}$.

M2. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$,

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(\mathbf{y}, \mathbf{x}),$$

and thus d is symmetric.

M3. For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$,

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) &= |x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| \\ &\leq |x_1 - z_1| + |x_2 - z_2| \\ &= d(\mathbf{x}, \mathbf{z}), \end{aligned}$$

where the usual triangle inequality on $\mathbb{R}$ has been applied twice. Hence d satisfies the triangle inequality.

(b) The neighborhood of radius 1 at $(0,0)$ is given by

$$N_1((0,0)) = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, (0,0)) < 1\} = \{\mathbf{x} \in \mathbb{R}^2 : |x_1| + |x_2| < 1\}$$

Consider the quadrant where $x_1 \geq 0$ and $x_2 \geq 0$. Then $|x_1| + |x_2| = x_1 + x_2$, and thus the boundary of the neighborhood is given by $x_2 = 1 - x_1$. This is a straight line which intercepts the x_2 axis at 1, and has slope -1. By symmetry, it follows that the neighborhood is diamond, with corners at $(0,1)$, $(0,1)$, $(0,-1)$, and $(-1,0)$. It is plotted in Fig. 1.

10. Suppose that f is not constant. Then there exists $x < y$ such that $f(x) \neq f(y)$. For an arbitrary $n \in \mathbb{N}$ consider the points

$$t_k = x - \frac{(y-x)k}{n}$$

for $k = 0, \ldots, n$. Then by the triangle inequality,

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \\ &\leq \sum_{k=1}^{n} (t_k - t_{k-1})^2 \\ &= \sum_{k=1}^{n} \left(\frac{y-x}{n}\right)^2 \\ &= \frac{(y-x)^2}{n}. \end{aligned}$$

Since n is arbitrary, there exists an n such that

$$n > \frac{(y-x)^2}{|f(x) - f(y)|}.$$

which leads to a contradiction. An alternative approach is to first note that for all $x, y \in \mathbb{R}$

$$\left|\frac{f(y) - f(x)}{y - x}\right| \leq \left|\frac{(y-x)^2}{y-x}\right| = |y - x|.$$

Since $|y-x| \to 0$ as $y \to x$, it follows that

$$\left|\frac{f(y)-f(x)}{y-x}\right|$$

as $y \to x$, and hence

$$\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = 0$$

for all $x \in \mathbb{R}$. Thus f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$. For any bounded interval $I \subseteq \mathbb{R}$, f must be constant. Hence f is constant on $\mathbb{R}$.

11. Suppose that f is differentiable on $\mathbb{R}$, and that $2 \leq f'(x) \leq 3$ for $x \in \mathbb{R}$. If $f(0) = 0$, prove that $2x \leq f(x) \leq 3x$ for all $x \geq 0$. Suppose $x > 0$. By the Mean Value Theorem, there exists a $y \in (0, x)$ such that

$$\frac{f(x)-f(0)}{x-0} = f'(y)$$

and hence

$$\frac{f(x)}{x} = f'(y).$$

The constraint on the derivative shows that

$$2 \leq \frac{f(x)}{x} \leq 3$$

and hence

$$2x \leq f(x) \leq 3x. \tag{1}$$

Since $f(0) = 0$, it follows that Eq. 1 holds for all $x \geq 0$.

12. Suppose that f is integrable on $[a, b]$. Then for all $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

Consider the partition Q composed of the values a, c, d, and b; in general this will have four points, but if $c = a$ or $d = b$ it may have fewer. Define a new partition $T = P \cup Q$. Then since T is a refinement,

$$U(f, T) - L(f, T) < \epsilon.$$

Write $T = \{a = t_0 < t_1 < \ldots < t_n = b\}$. There exists i and j such that $t_i = c$ and $t_j = d$. Consider the partition $S = \{t_i < \ldots < t_j\}$ of $[c, d]$. Then

$$\begin{aligned} U(f,S) - L(f,S) &= \sum_{k=i+1}^{j} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])) \\ &\leq \sum_{k=1}^{n} (M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])) \\ &= U(f,T) - L(f,T) \\ &< \epsilon. \end{aligned}$$

Since this is true for an arbitrary $\epsilon > 0$, it follows that f is integrable on $[c, d]$.

13. (a) Suppose that $r^{1/3}$ is rational. Then

$$r^{1/3} = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$. Hence

$$r = \frac{p^3}{q^3}$$

and thus r is rational. Thus is r is irrational, then $r^{1/3}$ is irrational also. Similarly, if $r + 1$ is rational, then

$$r + 1 = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$, where $q \neq 0$, and

$$r = \frac{p}{q} - 1 = \frac{p - q}{q}$$

so r is rational. Thus if r is irrational, then $r + 1$ is irrational also.

(b) First, consider the number $x = 5 + \sqrt{2}$. Then

$$\begin{aligned} x - 5 &= \sqrt{2} \\ (x-5)^2 &= 2 \\ x^2 - 10x + 25 &= 2 \\ x^2 - 10x + 23 &= 0. \end{aligned}$$

By the rational zeroes theorem, if x is rational, then $x = \pm 1$ or $x = \pm 23$. But $x > 5$ and $x < 5 + 2 = 7$, so none of these possibilities are valid, and thus x is irrational. By the results of part (a), $(5 + \sqrt{2})^{1/3}$ is irrational, and $(5 + \sqrt{2})^{1/3} + 1$ is irrational also.

14. For the first limit, L'Hôpital's rule can be applied once to show that

$$\lim_{x \to 0} \frac{x}{1 - e^{-x^2 - 3x}} = \lim_{x \to 0} \frac{1}{(2x+3)e^{-x^2-3x}} = \frac{1}{3}.$$

The second limit can first be rewritten as

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}$$

after which L'Hôpital's rule can be applied twice to show that

$$\lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\sin x}{2 \cos x + x \sin x} = \frac{0}{2} = 0.$$

For the third limit, L'Hôpital's rule can be applied three times to show that

$$\begin{aligned}\lim_{x\to 0}\frac{x^3}{\sin x - x} &= \lim_{x\to 0}\frac{3x^2}{\cos x - 1}\\ &= \lim_{x\to 0}\frac{6x}{-\sin x}\\ &= \lim_{x\to 0}\frac{6}{-\cos x}\\ &= \frac{6}{-1} = -6.\end{aligned}$$

15. Suppose $a > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-2N} < a$. Hence for $M > 2N$,

$$\sup\{s_n : n > M\} = a$$

and hence $\limsup s_n = a$. Now consider any $l > 0$. Then there exists an $N \in \mathbb{N}$ such that $2^{-n} < l$ for $n > 2N$. Hence l is not a lower bound for the set $\{s_n : n > M\}$ for any M. However, since 0 is lower bound, it follows that it must be the greatest lower bound, and hence $\limsup s_n = 0$.

Now suppose $a \leq 0$. Then $\inf\{s_n : n > M\} = a$ for all $M \in \mathbb{N}$, so $\liminf s_n = a$. Also,

$$\sup\{s_n : n > M\} = \begin{cases} 2^{-M-1} & \text{if } M \text{ is odd,}\\ 2^{-M-2} & \text{if } M \text{ is even.}\end{cases}$$

so $\limsup s_n = 0$.

If $a = 0$ then $\limsup s_n = \liminf s_n$ and hence the series converges with limit zero. Otherwise, $\limsup s_n \neq \liminf s_n$ and the series does not converge.

16. Consider any $\epsilon > 0$. Then

$$\begin{aligned}|f(x_1,x_2) - f(0,0)| &= \left|\frac{1}{x_1^2+x_2^2+1} - \frac{1}{1}\right|\\ &= \left|\frac{x_1^2+x_2^2}{x_1^2+x_2^2+1}\right|\\ &\leq |x_1^2+x_2^2|.\end{aligned}$$

Suppose $d((x_1,x_2),(0,0)) < \delta$ where $\delta = \sqrt{\epsilon}$. Then

$$|f(x_1,x_2) - f(0,0)| \leq (d((x_1,x_2),(0,0)))^2 < \delta^2 = \epsilon.$$

Now consider the point $(0,1)$:

$$\begin{aligned} |f(x_1,x_2) - f(0,1)| &= \left| \frac{1}{x_1^2 + x_2^2 + 1} - \frac{1}{2} \right| \\ &= \left| \frac{x_1^2 + x_2^2 - 1}{2(x_1^2 + x_2^2 + 1)} \right| \\ &\leq \frac{|x_1^2 + x_2^2 - 1|}{2} \\ &= \frac{|x_1^2 + (x_2 - 1)(x_2 + 1)|}{2} \\ &\leq \frac{|x_1|^2 + |x_2 - 1| \cdot |x_2 + 1|}{2}. \end{aligned}$$

Suppose $d((x_1,x_2),(0,1)) < 1$, so that

$$x_1^2 + (x_2 - 1)^2 < 1.$$

from which it follows that $|x_1| < 1$ and $|x_2 - 1| < 1$. Hence $|x_2 + 1| \leq |x_2| + 1 \leq 3$. Define $\delta = \min\{1, \frac{\epsilon}{3}\}$. Then $\delta < \frac{\epsilon}{3}$ and $\delta < \sqrt{\epsilon}$. Hence $|x_1| < \sqrt{\epsilon}$ and $|x_2 - 1| < \frac{\epsilon}{3}$. Then

$$|f(x_1,x_2) - f(0,1)| \leq \frac{|x_1|^2 + |x_2 - 1| \cdot |x_2 + 1|}{2} < \frac{(\sqrt{\epsilon})^2 + 3\frac{\epsilon}{3}}{2} = \epsilon.$$

Hence f is continuous at $(0,1)$.

17. (a) For $p \neq 1$, the improper integral can be written as

$$\begin{aligned} \int_0^1 x^{-p}\,dx &= \lim_{c \to 0^+} \int_c^1 x^{-p}\,dx \\ &= \lim_{c \to 0^+} \left[\frac{x^{-p+1}}{-p+1} \right]_c^1 \\ &= \lim_{c \to 0^+} \frac{1 - c^{1-p}}{1 - p}. \end{aligned}$$

Then if $0 < p < 1$, the exponent is positive, so

$$\int_0^1 x^{-p}\,dx = \frac{1}{1-p}$$

whereas if $p > 1$, the exponent is negative, so

$$\int_0^1 x^{-p}\,dx = \infty.$$

(b) The improper integral can be written as

$$\int_0^1 x^{-p}\,dx = \lim_{c\to 0^+}\int_c^1 x^{-p}\,dx + \lim_{d\to\infty}\int_1^d x^{-p}\,dx.$$

Since both terms are non-negative, it follows that if one term is ∞, then the integral must ∞ also. By the result above, if $p > 1$, then the first term is ∞. Now suppose $p = 1$. Then the second term is

$$\begin{aligned}\lim_{d\to\infty}\int_1^d x^{-1}\,dx &= \lim_{d\to\infty}[\log x]_1^d \\ &= \lim_{d\to\infty}\log d \\ &= \infty.\end{aligned}$$

If $0 < p < 1$, then second term is

$$\begin{aligned}\lim_{d\to\infty}\int_1^d x^{-p}\,dx &= \lim_{d\to\infty}\left[\frac{x^{-p+1}}{1-p}\right]_1^d \\ &= \lim_{d\to\infty}\frac{d^{-p+1}-1}{1-p} \\ &= \infty.\end{aligned}$$

Hence in all cases, $\int_0^1 x^{-p}dx = \infty$.

18. Since f is integrable on $[a,b]$, it is bounded, so there exists a $B > 0$ such that $f(x) < B$ for all $x \in [a,b]$. Assume that if f is integrable on $[a,b]$ then it is integrable on any interval $[c,d] \subseteq [a,b]$; for full details see Problem 12.

To show the above limit, consider any $\epsilon > 0$, and examine $c \in (b-\delta, b)$ where $\delta = \epsilon/B$. Then

$$\begin{aligned}\left|\int_a^c f - \int_a^b f\right| &= \left|\int_c^b f\right| \\ &\le \int_c^b |f| \\ &\le (b-c)B \\ &< \delta B = \epsilon.\end{aligned}$$

Hence

$$\lim_{d\to b^-}\int_a^d f(x)\,dx = \int_a^b f(x)\,dx.$$

19. If $\lambda = 0$, then $s_n = 0$ for all n, so s_n is a constant convergent sequence. Similarly if $\lambda = 1$, then $s_n = 1$ for all n, so s_n is also constant and convergent. In general,

$$s_n = \lambda^{2^{n-1}}.$$

If $0 < \lambda < 1$, then since $0 < \lambda^{2^{n-1}} \leq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \to 0$ as $n \to \infty$ it follows that $s_n \to 0$. If $\lambda > 1$, then since $\lambda^{2^{n-1}} \geq \lambda^n$ for $n \in \mathbb{N}$ and $\lambda^n \to \infty$ as $n \to \infty$, it follows that $s_n \to \infty$.

Finally, suppose $\lambda < 0$. Then for $n \geq 2$, the resulting sequence (s_n) will be the same as the case for $-\lambda$, and will therefore have the same convergence properties. Hence (s_n) converges if and only if $|\lambda| \leq 1$.

20. The minimum element of $[0, \sqrt{2}]$ is 0, and since this is also in A it follows that $\min A = 0$. The maximum element of $[0, \sqrt{2}]$ is $\sqrt{2}$, but this is not in A. Since there are rational numbers arbitrarily close to $\sqrt{2}$, it follows that A does not have a maximum. The infimum is just $\inf A = \min A = 0$. $\sqrt{2}$ is an upper bound for A. For any $\epsilon > 0$, there exist elements in A which are greater than $\sqrt{2} - \epsilon$, and thus $\sqrt{2}$ is the least upper bound. Hence $\sup A = \sqrt{2}$.

For B, note that

$$x^2 + x - 1 = (x + 1/2)^2 - 5/4.$$

Since the first term can take any positive value, it follows that $B = [-5/4, \infty)$. Hence $\min B = \inf B = -5/4$, the maximum does not exist, and $\sup B = \infty$.

By completing the square, above equation can be written as

$$x^2 + x - 1 = \left(x + \frac{1 - \sqrt{5}}{2}\right)\left(x + \frac{1 + \sqrt{5}}{2}\right).$$

The quadratic will be strictly negative when one of these two factors is strictly negative and the other is strictly positive. Hence $C = (-1/2 - \sqrt{5}/2, -1/2 + \sqrt{5}/2)$, so the minimum and maximum do not exist, $\inf C = -1/2 - \sqrt{5}/2$ and $\sup C = -1/2 + \sqrt{5}/2$.

21. (a) Suppose $0 \leq x < 1$. Then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (x - x^n) = x$$

since if $|x| < 1$, then $x^n \to 0$ as $n \to \infty$. For $x = 1$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (1 - 1^n) = 0.$$

Hence f_n converges pointwise to a limit f on $[0, 1]$ given by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Graphs of several of the f_n and the limit f are shown in Fig. 2.

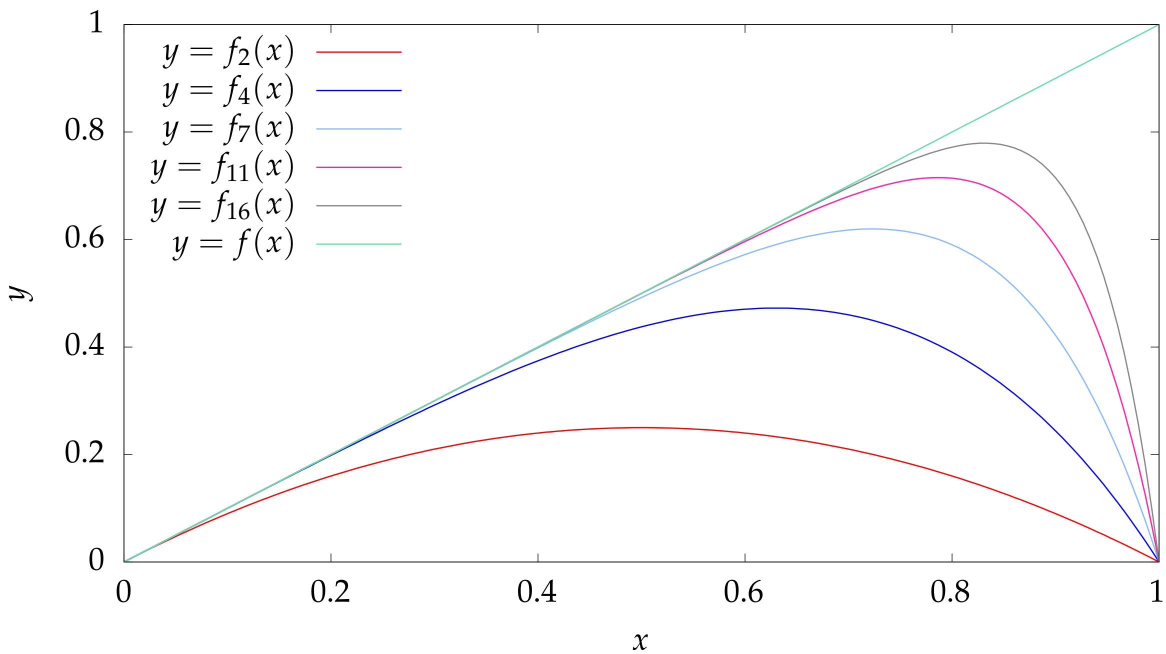

Figure 2: Graph for Problem 21, on pointwise and uniform convergence.

(b) As can be seen from Fig. 2, the convergence does not appear to be uniform on $[0,1]$, since it does not appear that the f_n will ever lie within a strip of a fixed width ϵ around f. To see this mathematically, for a given n, consider the point $x = (1/2)^{1/n}$. Then

$$\begin{aligned} |f_n(x) - f(x)| &= |x - x^n - x| \\ &= |x^n| \\ &= 1/2 \end{aligned}$$

Hence if $\epsilon = 1/2$, there does not exist an N such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [0,1]$.

(c) Consider the interval $[0, 1/2]$. Then for any x in this interval

$$|f_n(x) - f(x)| = |x^n| \leq 2^{-n}$$

Consider any $\epsilon > 0$. Then there exists an N such that $n > N$ implies $2^{-n} < \epsilon$, and thus $|f_n(x) - f(x)| < \epsilon$. Hence $f_n \to f$ uniformly on $[0, 1/2]$.

(d) Since continuous functions are integrable, it follows immediately that f_n is integrable for all $n \in \mathbb{N}$. For a specific n,

$$\begin{aligned} \int_0^1 f_n &= \int_0^1 (x - x^n)dx \\ &= \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{n+1} \\ &= \frac{n-1}{2(n+1)}. \end{aligned}$$

To show that f is integrable, consider the function

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Choose any $\epsilon > 0$, and examine the partition $P = \{0 = t_0 < t_1 < t_2 = 1\}$ where $t_1 = 1 - \epsilon/2$. Then

$$L(f,P) = \sum_{k=1}^{2} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) = \sum_{k=1}^{2} 0 = 0$$

and

$$U(f,P) = \sum_{k=1}^{2} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) = (t_1 - t_0) \cdot 0 + (t_2 - t_1)1 = \frac{\epsilon}{2}.$$

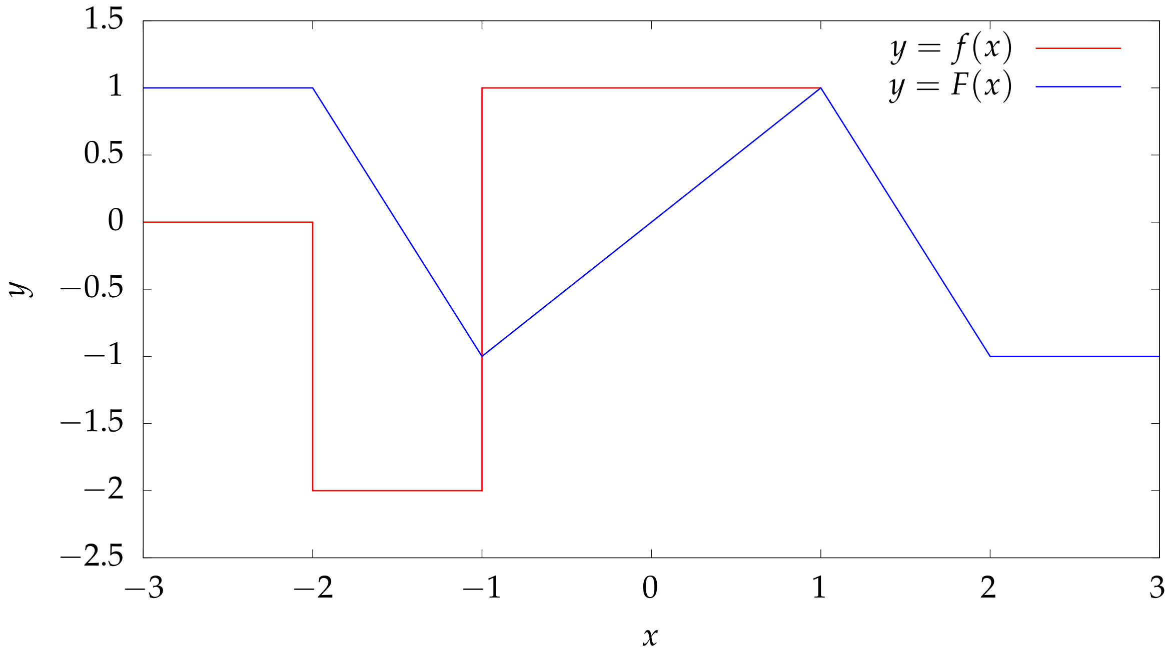

Figure 3: Graph for Problem 22, on the second Fundamental Theorem of Calculus.

Thus $U(f,P) - L(f,P) < \epsilon$. Since a partition such as this can be constructed for an arbitrary $\epsilon > 0$, it follows that g is integrable and $\int_0^1 g = 0$. Since $f(x) = x - g(x)$ and both x and g are integrable, it follows that f is integrable and $\int_0^1 f = \int_0^1 x - \int_0^1 g = 1/2$. Note that

$$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \frac{n-1}{2(n+1)} = \frac{1}{2} = \int_0^1 f.$$

22. (a) If $0 \le x \le 1$, then

$$F(x) = \int_0^x f = \int_0^x 1\,dt = x.$$

If $1 < x \le 2$, then

$$F(x) = \int_0^1 f + \int_1^x = 1 - 2(x-1) = 3 - 2x.$$

If $x > 2$, then

$$F(x) = F(2) + \int_2^x 0 = F(2) = -1.$$

For negative values of x, note that f is an even function, and thus

$$F(-x) = \int_0^{-x} f(t)dt = \int_0^x f(-s)(-ds) = -\int_0^x f(s)ds = -F(x)$$

so F is odd. Hence

$$F(x) = \begin{cases} 1 & \text{if } x < -2, \\ -3-2x & \text{if } -2 \leq x < 1, \\ x & \text{if } -1 \leq x \leq 1, \\ 3-2x & \text{if } 1 < x \leq 2, \\ -1 & \text{if } x > 2. \end{cases}$$

(b) The functions f and F are plotted in Fig. 3.

(c) By the second Fundamental Theorem of Calculus, if f is continuous at x, and then F is differentiable at x and $F'(x) = f(x)$. Thus the only points where F may not be defined are $x = \pm 1, \pm 2$. Since

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x-1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$$

but

$$\lim_{x \to 1^+} \frac{F(x) - F(1)}{x-1} = \lim_{x \to 1^-} \frac{3-2x-1}{x-1} = -2$$

so F is not differentiable at 1. Similarly

$$\lim_{x \to 2^-} \frac{F(x) - F(2)}{x-2} = \lim_{x \to 2^-} \frac{3-2x-(-1)}{x-2} = 1$$

but

$$\lim_{x \to 2^+} \frac{F(x) - F(2)}{x-2} = \lim_{x \to 2^+} \frac{0}{x-2} = 0$$

so F is not differentiable at 2. Since F is odd, it follows that F is not differentiable at -1 and -2 also. Hence F' is defined on $\mathbb{R}/\{-2,-1,1,2\}$ and

$$F(x) = \begin{cases} 0 & \text{if } x < -2, \\ -2 & \text{if } -2 < x < 1, \\ 1 & \text{if } -1 < x < 1, \\ -2 & \text{if } 1 < x < 2, \\ 0 & \text{if } x > 2. \end{cases}$$

23. (a) Let f and g be continuous functions on $[a,b]$ such that $\int_a^b f = \int_a^b g$. Prove that there exists an $x \in [a,b]$ such that $f(x) = g(x)$. If $\int_a^b f = \int_a^b g$, then if $h(x) =$

$f(x) - g(x)$, then $\int_a^b h = 0$. Consider the partition $P = \{a = t_0 < t_1 = b\}$. Then

$$0 = \int_a^b h \le U(h, P) = (b - a)M(h, [a, b])$$

and

$$0 = \int_a^b h \ge L(h, P) = (b - a)m(h, [a, b]).$$

Since a continuous function on a closed interval achieves its bounds, there exist x_1 and x_2 such that $h(x_1) = M(h, [a, b])$ and $h(x_2) = m(h, [a, b])$. Either $h(x_1) = 0$ or $h(x_2) = 0$, or otherwise $h(x_1) > 0$ and $h(x_2) < 0$. In the latter case, the intermediate value theorem can be applied to show that there exists an x_3 between x_1 and x_2 such that $h(x_3) = 0$. In all cases there exists an x such that $h(x) = 0$ and hence $f(x) = g(x)$.

(b) On the interval $[-1, 1]$, define

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \ge 0, \end{cases}$$

and let $g(x) = -f(x)$. By construction $f(x) \ne g(x)$ for all $x \in [-1, 1]$. To find the integral of f, choose $\epsilon > 0$ and consider the partition $P = \{a = -1 < t_1 < t_2 < t_3 = 1\}$ where $t_1 = -\epsilon/5$ and $t_2 = \epsilon/5$. Then

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{3} (t_k - t_{k-1}) m(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(-1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= -\frac{2\epsilon}{5}. \end{aligned}$$

Similarly

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{3} (t_k - t_{k-1}) M(f, [t_{k-1}, t_k]) \\ &= \left(1 - \frac{\epsilon}{5}\right)(-1) + \frac{2\epsilon(1)}{5} + \left(1 - \frac{\epsilon}{5}\right)(1) \\ &= \frac{2\epsilon}{5}. \end{aligned}$$

Then $U(f, P) - L(f, P) = 4\epsilon/5 < \epsilon$, and since ϵ is arbitrary it follows that f is integrable, and that $\int_{-1}^1 f = 0$. In addition, so $\int_{-1}^1 g = \int_{-1}^1 (-f) = -\int_{-1}^1 f = 0$. Thus $\int_{-1}^1 f = \int_{-1}^1 g$ but $f(x) \ne g(x)$ for all $x \in [-1, 1]$.

24. (a) The functions h_1, h_2, and h_3 are plotted in Fig. 4

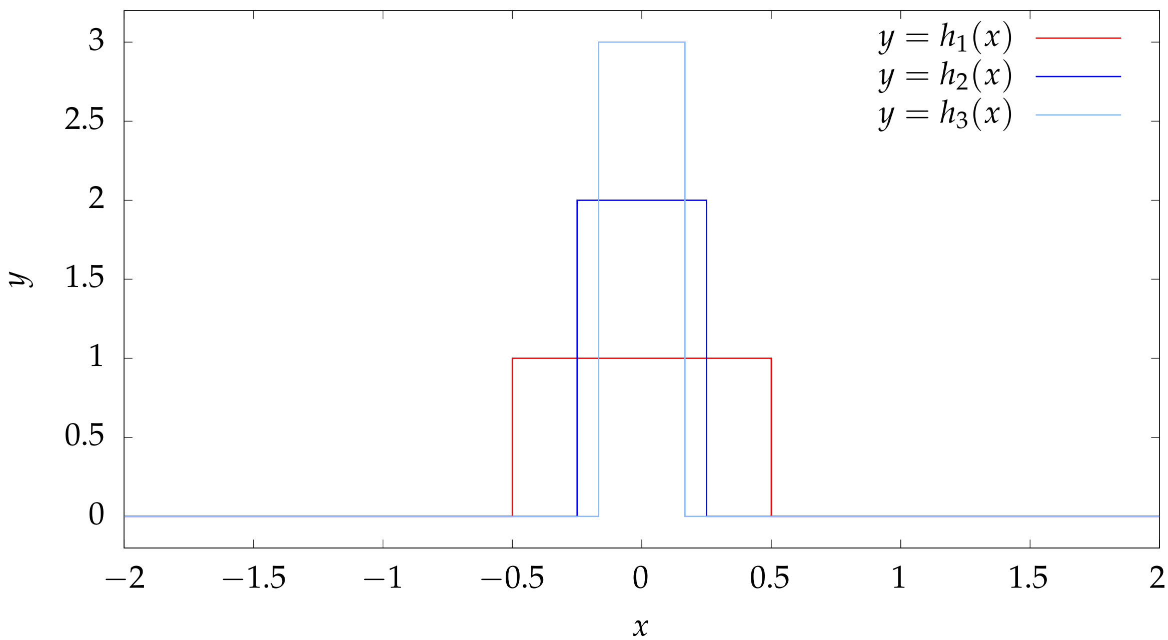

Figure 4: Graphs of several functions $h_n(x)$ used in Problem 24 on integration limits.

(b) Consider any $x \neq 0$, and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $1/N < 2|x|$. Then if $n > N$, $h_n(x) = 0$. Hence $\lim_{n\to\infty} h_n(x) = 0$. At $x = 0$,

$$h_n(x) = n$$

which tends to ∞ as $n \to \infty$.

(c) Consider any $\epsilon > 0$. Then since f is continuous, there exists a $\delta > 0$ such that $|x| < \delta$ implies $|f(x) - f(0)| < \epsilon/2$. Then there exists an N such that $1/2N < \delta$. By using the definition of an improper integral,

$$\begin{aligned}\int_{-\infty}^{\infty} h_n f &= \lim_{a\to-\infty} \int_a^0 h_n f + \lim_{b\to\infty} \int_0^b h_n f \\ &= \int_{-1/2n}^{0} nf + \int_0^{1/2n} nf \\ &= n \int_{-1/2n}^{1/2n} f.\end{aligned}$$

By using the bound on f due to continuity,

$$n \int_{-1/2n}^{1/2n} \left(f(0) - \frac{\epsilon}{2}\right) dx \leq \int_{-\infty}^{\infty} h_n f \leq n \int_{-1/2n}^{1/2n} \left(f(0) + \frac{\epsilon}{2}\right) dx$$

so

$$f(0) - \frac{\epsilon}{2} \leq \int_{-\infty}^{\infty} h_n f \leq f(0) + \frac{\epsilon}{2}.$$

Hence

$$\left| f(0) - \int_{-\infty}^{\infty} h_n f \right| \leq \frac{\epsilon}{2} < \epsilon$$

so

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n f = f(0).$$

(d) Consider

$$g(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Then for any $n \in \mathbb{N}$,

$$\int_{\infty}^{\infty} h_n g = \int_{-1/2n}^{1/2n} ng = \int_{-1/2n}^{0} 0 + \int_0^{1/2n} n = \frac{1}{2}$$

and hence

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n g = \frac{1}{2}$$

which does not equal $g(0) = 1$.

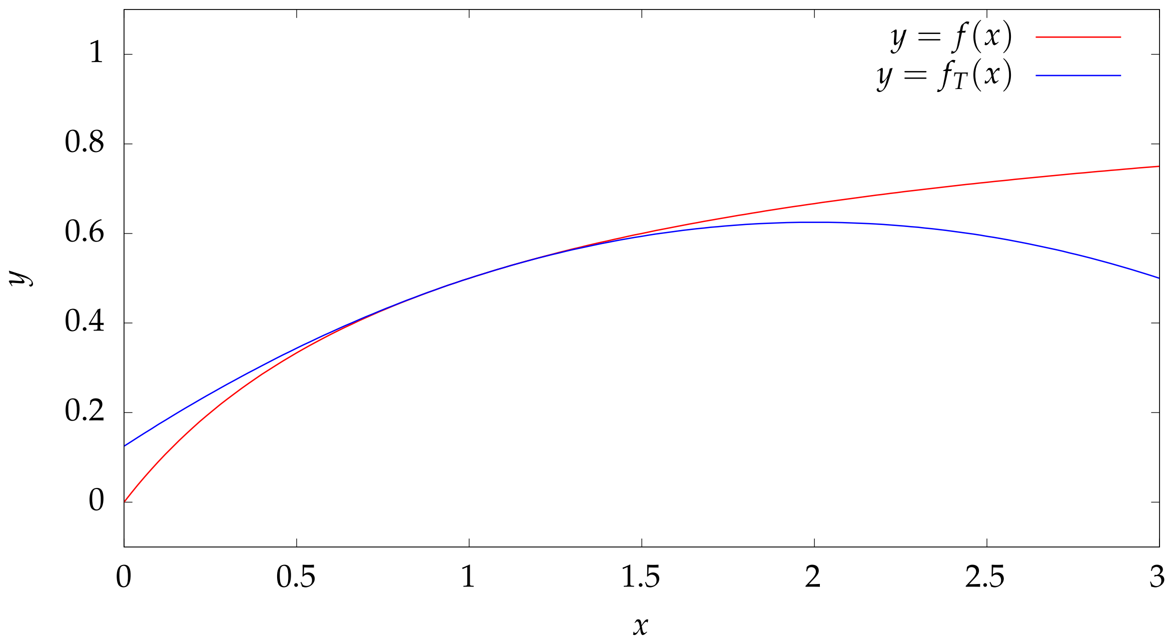

Figure 5: A graph of the function f and a Taylor series approximation f_T at $x = 1$, discussed in Problem 25.

25. (a) For $x > 0$,

$$f(x) = \frac{1}{1 + \frac{1}{x}}$$

and since $1/x \to 0$ as $x \to \infty$, it follows that $f(x) \to 0$ as $x \to \infty$. On the interval under consideration $0 \le x < 1 + x$, so $0 \le f(x) < 1$.

(b) The function f is shown in Fig. 5.

(c) Since

$$f(x) = 1 - \frac{1}{x+1},$$

it follows that the derivatives are

$$f'(x) = \frac{1}{(x+1)^2}$$

and

$$f''(x) = -\frac{2}{(x+1)^3}.$$

Hence $f(1) = 1/2$, $f'(1) = 1/4$, $f''(1) = -1/4$ and thus

$$\begin{aligned} f_T(x) &= \sum_{n=0}^{2} \frac{(x-1)^n f^{(n)}(1)}{n!} \\ &= f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2} \\ &= \frac{1}{2} + \frac{x-1}{4} - \frac{(x-1)^2}{8}. \end{aligned}$$

(d) f_T can rewritten as

$$\begin{aligned} f_T(x) &= \frac{1}{2} + \frac{x}{4} - \frac{1}{4} - \frac{x^2}{8} + \frac{x}{4} - \frac{1}{8}. \\ &= \frac{1}{8} + \frac{x}{2} - \frac{x^2}{8} \end{aligned}$$

which is a quadratic.

(e) The function f_T is shown in Fig. 5. By construction, the curves intersect at $x = 1$, and have the same slope and curvature there.

26. For the first series,

$$\beta = \limsup |a_n|^{1/n} = \limsup \left| \frac{a_{n+1}}{a_n} \right| = \limsup \sqrt{\frac{n^2+1}{(n+1)^2+1}} = 1,$$

so the radius of convergence is $R = 1$. At $x = -1$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$$

which converges by the alternating series theorem. At $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}.$$

Since $\sqrt{n^2+1} \le \sqrt{n^2+2n+1} = n+1$ and $1/(n+1)$ diverges, it follows that the series diverges at $x = 1$. Hence the exact interval of convergence is $[-1, 1)$.

For the second series, by looking at the limit of positive coefficients,

$$\beta = \limsup |a_n|^{1/n} = \lim_{n\to\infty} |a_{2n}|^{1/2n} = \lim_{n\to\infty} \left| \frac{(-2)^n}{n^2} \right|^{1/2n} = \lim_{n\to\infty} (\sqrt{2}) n^{1/n} = \sqrt{2},$$

and thus the radius of convergence is $R = 2^{-1/2}$. At $x = R$, the series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which converges by the alternating series theorem. Since the series is even, the sum converges at $x = -R$ also. Hence the interval of convergence is $[-2^{-1/2}, 2^{-1/2}]$.

27. If a sequence a_n converges to a limit a, then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|a_n - a| < \epsilon$.

Consider any $\epsilon > 0$. Since $s_n \to s$, there exists an N_1 such that $n > N_1$ implies that

$$|s_n - s| < \frac{\epsilon}{4}$$

and since $t_n \to t$, there exists an N_2 such that $n > N_2$ implies that

$$|t_n - t| < \frac{\epsilon}{4}.$$

Define $N = \max\{N_1, N_2\}$. Then for $n > N$,

$$\begin{aligned} |(3s_n + t_n) - 3s - t| &\le 3|s_n - s| + |t_n - t| \\ &< \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon \end{aligned}$$

and thus $3s_n + t_n \to 3s + t$ as $n \to \infty$.

28. (a) By making use of the definition,

$$\varphi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{6+2\sqrt{5}}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1+\varphi.$$

(b) First consider the induction hypothesis H_1:

$$f(0) = \frac{\varphi^0 - (1-\varphi)^0}{\sqrt{5}} = 0, \qquad f(1) = \frac{\varphi - (1-\varphi)}{\sqrt{5}} = \frac{2\varphi - 1}{\sqrt{5}} = 1.$$

Now consider the induction step. Suppose H_n is true, and consider H_{n+1}. Then $F_n = f(n)$ and

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} = \frac{\varphi^n - (1-\varphi)^n + \varphi^{n-1} - (1-\varphi)^{n-1}}{\sqrt{5}} \\ &= \frac{\varphi^{n-1}(1+\varphi) - (1-\varphi)^{n-1}(1+(1-\varphi))}{\sqrt{5}}. \end{aligned}$$

Since $(1-\varphi)^2 = 1 - 2\varphi + \varphi^2 = 1 - 2\varphi + (1+\varphi) = 2 - \varphi$, it follows that

$$F_{n+1} = \frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\sqrt{5}} = f(n+1)$$

so H_{n+1} is true. Hence by mathematical induction H_n is true for all n, so $F_n = f(n)$ for all $n \in \mathbb{N} \cup \{0\}$.

(c) By using the explicit formula for F_n,

$$\frac{F_{n+1}}{F_n} = \frac{\varphi^{n+1} - (1-\varphi)^{n+1}}{\varphi^n - (1-\varphi)^n} = \varphi \frac{1 - \left(\frac{1-\varphi}{\varphi}\right)^{n+1} \varphi^{-1}}{1 - \left(\frac{1-\varphi}{\varphi}\right)^n}.$$

Since $|\varphi| > 1$ and $|1-\varphi| = \frac{\sqrt{5}-1}{2} < 1$, then

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi \frac{1-0}{1-0} = \varphi.$$

29. (a) Define $\alpha = \sup S$. Since α is an upper bound for S, then $\alpha \geq s$ for all $s \in S$. Hence $f(\alpha) \geq f(s)$ for all $s \in S$, so $f(\alpha) \geq t$ for all $t \in T$, so $f(\alpha)$ is an upper bound for T.

Since f is a continuous strictly increasing function there is an inverse f^{-1}. Suppose $\beta < f(\alpha)$ is an upper bound for T. Then $f^{-1}(\beta) < \alpha$, and since α is the supremum of S, there exists an $s \in S$ such that $s > f^{-1}(\beta)$. But then $f(s) > \beta$, so there exists a $t \in T$ such that $t > \beta$ which is a contradiction. Hence $f(\alpha)$ is the least upper bound, so $\sup B = f(\sup A)$.

(b) This follows from the result in part (a) and the fact that f is continuous:

$$\begin{aligned}\limsup b_n &= \lim_{N\to\infty}(\sup\{b_n : n > N\}) \\ &= \lim_{N\to\infty}(\sup\{f(a_n) : n > N\}) \\ &= \lim_{N\to\infty} f(\sup\{a_n : n > N\}) \\ &= f\left(\lim_{N\to\infty}\sup\{a_n : n > N\}\right) \\ &= f(\limsup a_n).\end{aligned}$$

(c) Let $S = (-1, 0)$ and

$$f(x) = \begin{cases} x & \text{if } x < 0, \\ x+1 & \text{if } x \geq 0. \end{cases}$$

Then $\sup S = 0$, so $f(\sup S) = 1$. However, $T = (-1, 0)$ and $\sup T = 0 \neq 1$.

30. (a) $d((0,1),(0,0)) = \min\{0,2\} = 0$, which violates the property that $d(\mathbf{x},\mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

(b) Consider the three properties of a metric:

M1. Note that $d(\mathbf{x},\mathbf{x}) = \max\{0,0\} = 0$. If $d(\mathbf{x},\mathbf{y}) = 0$, then $\max\{|x_1 - y_1|, 2|x_2 - y_2|\} = 0$ so $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$, and $x_1 = y_1$ and $x_2 = y_2$, so $\mathbf{x} = \mathbf{y}$.

M2. $d(\mathbf{x},\mathbf{y}) = \max\{|x_1 - y_1|, 2|x_2 - y_2|\} = \max\{|y_1 - x_1|, 2|y_2 - x_2|\} = d(\mathbf{y},\mathbf{x})$.

M3. For any $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}$,

$$\begin{aligned}d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) &= \max\{|x_1 - y_1|, 2|x_2 - y_2|\} + \max\{|y_1 - z_1|, 2|y_2 - z_2|\} \\ &\geq \max\{|x_1 - y_1| + |y_1 - z_1|, 2|x_2 - y_2| + 2|y_2 - z_2|\} \\ &\geq \max\{|x_1 - z_1|, 2|x_2 - z_2|\}\end{aligned}$$

where on the final line, the usual triangle inequality has been applied.

Hence d_B is a metric. The neighborhood of radius 1 at $(0,0)$ is shown in Fig. 6.

(c) By the given continuity property, for all $x \in X$, and all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(y,x) < \delta$ implies $d_E(f(x), f(y)) < \epsilon/2$, so that

$$\sqrt{(f(x_1) - f(y_1))^2 + (f(x_2) - f(y_2))^2} < \frac{\epsilon}{2}$$

and hence $|f(x_1) - f(y_1)| < \epsilon/2$ and $|f(x_2) - f(y_2)| < \epsilon/2$. Then

$$d_B(f(x), f(y)) = \max\{|f(x_1) - f(y_1)|, 2|f(x_2) - f(y_2)|\} < 2\frac{\epsilon}{2} = \epsilon.$$

Hence f is continuous with respect to (X, d) and $(\mathbb{R}^2, d_B)$ also.

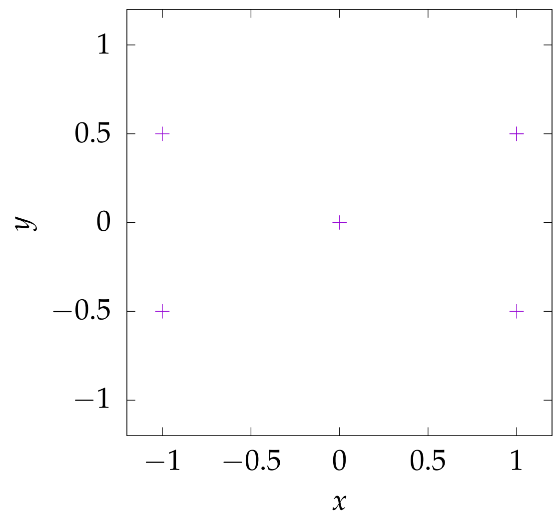

Figure 6: Graph of the neighborhood of radius 1 at $(0,0)$ with respect to d_B.

31. (a) For $x \in [0,1]$, $F(x) = \int_0^x 0 = 0$. For $x > 1$,

$$F(x) = \int_1^x f(t)dt = \left[\frac{t^2}{2}\right]_1^x = \frac{x^2-1}{2}.$$

(b) By the second Fundamental Theorem of Calculus, if f is continuous at a given x, then F is differentiable at x and $F'(x) = f(x)$. Hence the only value to check is $x = 1$ where f is not continuous:

$$\lim_{y \to 1^+} \frac{F(y) - F(1)}{y-1} = \lim_{y \to 1^+} \frac{y^2-1}{2(y-1)} = \lim_{y \to 1^+} \frac{y+1}{2} = 1$$

and

$$\lim_{y \to 1^-} \frac{F(y) - F(1)}{y-1} = 0$$

so F is not differentiable at 1. Hence F' exists on $[0,1) \cup (1,\infty)$, and

$$F'(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ x & \text{if } x > 1. \end{cases}$$

(c) Since $\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$, but $f(1) = 0$, f is not continuous, and hence it is not uniformly continuous. To show that F is not uniformly continuous, consider $\epsilon = 1/2$, and for a $\delta > 0$, put $\eta = \min\{\delta, 1\}$, and examine $x = \eta^{-1}$ and

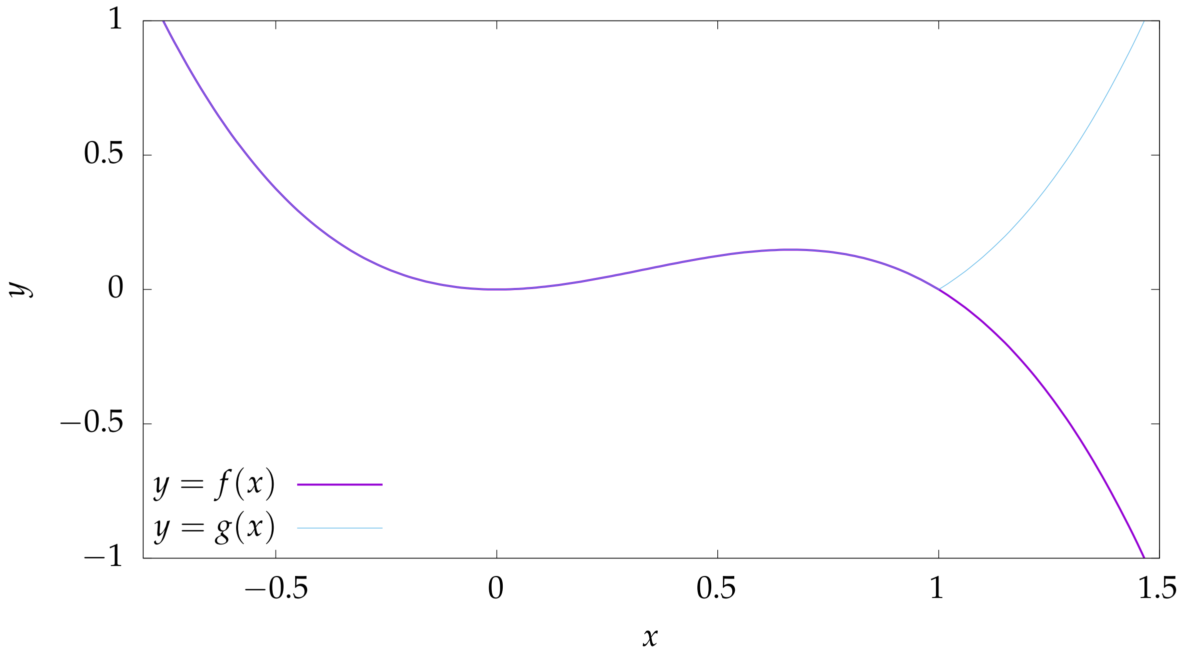

Figure 7: Graph of the functions considered in question 7 on Taylor series.

$y = \eta^{-1} + \eta/2$. Then $|x - y| < \delta$, but

$$|f(x) - f(y)| = \frac{|\eta^{-2} - \eta^{-2} - 1 - \frac{\eta^2}{2}|}{2} > \frac{1}{2}.$$

Since an x and y with this property can be found for any $\delta > 0$, it follows that F is not uniformly continuous.

32. (a) The functions are plotted in Fig. 7.

(b) Since

$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x^2(1-x)|}{x} = \lim_{x \to 0^+} x(1-x) = 0,$$

and

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x^2(1-x)|}{x} = \lim_{x \to 0^-} x(1-x) = 0,$$

so the two limits agree then g is differentiable at $x = 0$. However

$$\lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x^2(1-x)|}{x - 1} = \lim_{x \to 1^+} x^2 = 1,$$

and

$$\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{|x^2(1-x)|}{x-1} = \lim_{x\to 1^-} -x^2 = -1,$$

so g is not differentiable at $x = 1$.

(c) For $x \geq 1$, $g(x) = -f(x) = x^3 - x^2$, and the derivatives are $g'(x) = 3x^2 - 2x$, $g''(x) = 6x - 2$, $g'''(x) = 6$, and $g^{(n)}(x) = 0$ for $n \geq 4$. Hence, the Taylor series of g at $x = 2$ is

$$\begin{aligned} g(x) &= g(2) + g'(2)(x-2) + \frac{g''(2)}{2!}(x-2)^2 + \frac{g'''(2)}{3!}(x-2)^3 \\ &= 4 + 8(x-2) + 5(x-2)^2 + (x-2)^3 \\ &= 4 + 8x - 16 + 5x^2 - 20x + 20 + x^3 - 6x^2 + 12x - 8 \\ &= x^3 - x^2 \end{aligned}$$

which is equal to $-f(x)$ for all x.

33. (a) The functions are plotted in Fig. 8.

(b) Since $f_n(0) = 0$ for all n, then $\lim_{n\to\infty} f_n(0) = 0$. Consider any $x > 0$. Then there exists an $n \in \mathbb{N}$ such that $Nx > 1$. But then for all $n > N$, $f_n(x) = ng(nx) = 0$, so $\lim_{n\to\infty} f_n(x) = 0$. Hence f_n converges pointwise to f, where $f(x) = 0$ for all $x \in [0, 1]$.

(c) For any $n \in \mathbb{N}$,

$$|f_n(1/n) - f(1/n)| = |n - 0| = n \geq 1.$$

Hence there does not exist any $N \in \mathbb{N}$ such that $n > N$ implies that $|f_n(x) - f(x)| < 1$ for all $x \in [0, 1]$. Hence f_n does not converge uniformly to f.

(d) The integrals are

$$\int_0^1 f_n = \int_0^{1/n} n^2 x \, dx = \left[n^2 \frac{x^2}{2} \right]_0^{1/n} = \frac{1}{2}.$$

Since f is identically zero, $\int_0^1 f = 0$. Hence the integrals of f_n do not converge to f.

34. (a) Consider any $\epsilon > 0$. Since $f'(x) \to 0$ as $x \to \infty$, there exists an $M > 0$ such that for $x > M$, $|f'(x) - 0| < \epsilon$. Consider any $x > M$. By the Mean Value Theorem, there exists a $y \in (x, x+1)$ such that

$$\frac{f(x+1) - f(x)}{x + 1 - x} = f'(y).$$

Since $y > M$ it follows that

$$|g(x)| = |f(x+1) - f(x)| = |f'(y)| < \epsilon.$$

Hence $g(x) \to 0$ as $x \to \infty$.

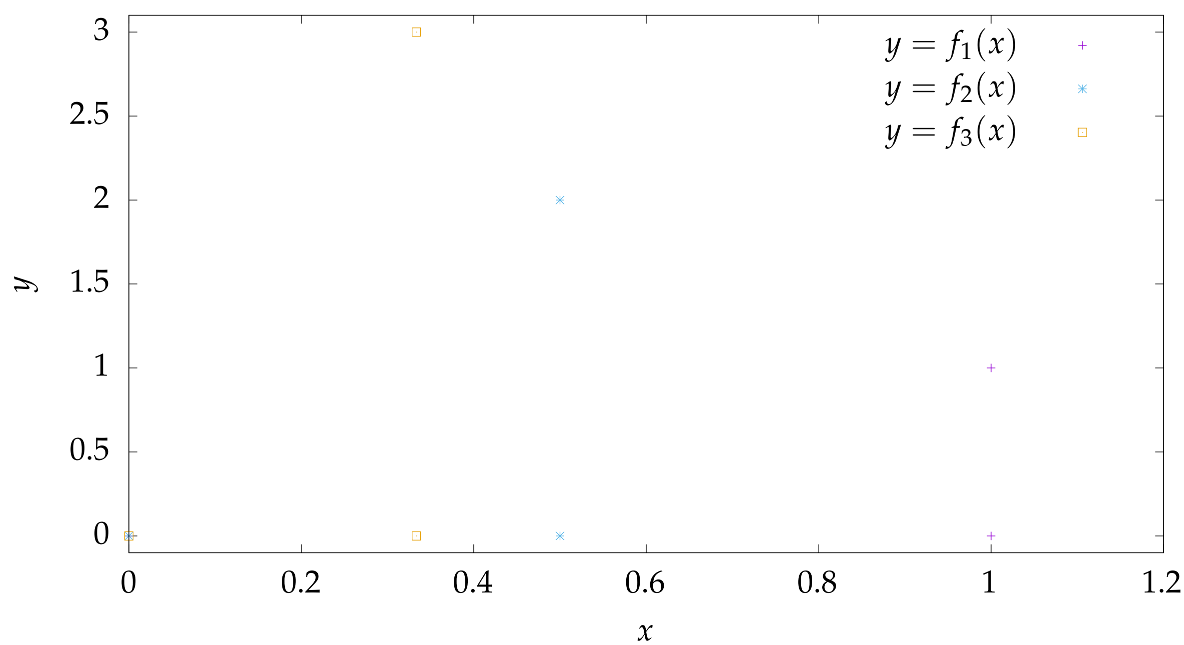

Figure 8: Graph of the functions considered in question 8 on pointwise and uniform convergence.

(b) For all $n \in \mathbb{N}$

$$p(-3) < 0, \qquad p(0) = 1, \qquad p(1) = -2, \qquad p(2) > 0.$$

The Intermediate Value Theorem can be applied to the intervals $(-3, 0)$, $(0, 1)$, and $(1, 2)$ to show that p has at least three roots.

By the rational zeroes theorem, any rational solution must have the form $x = \pm 1$. However, since $p(1) = -2$, and $p(-1) = 4$, it follows that the roots must be irrational.

35. (a) By applying L'Hôpital's rule once,

$$\lim_{x \to 0} \frac{x}{e^x - e^{-x}} = \lim_{x \to 0} \frac{1}{e^x + e^{-x}} = \frac{1}{2}.$$

By applying L'Hôpital's rule twice,

$$\lim_{x \to 0} \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \frac{2 \cos x \sin x}{2x} = \lim_{x \to 0} \frac{2 \cos^2 x - 2 \sin^2 x}{2} = 1.$$

(b) L'Hôpital's rule can be applied to show that

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{2h} \\ &\quad - \lim_{h \to 0} \frac{f'(x-h) - f'(x)}{2h} \\ &= \frac{f''(x) + f''(x)}{2} = f''(x). \end{aligned}$$

(c) Consider the function

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } -1 < x < 0, \end{cases}$$

defined on $(-1, 1)$. This is discontinous at 0, and thus it is not twice differentiable there.

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h \to 0} \frac{1 + (-1) - 0}{h^2} = \lim_{h \to 0} \frac{0}{h^2} = 0.$$

Any odd function that is not twice differentiable will give the same result.

36. (a) Consider any $\epsilon > 0$. Then there exists an N such that $|f_N(x) - f(x)| < \frac{\epsilon}{3(b-a)}$ for all $x \in [a, b]$. Since f_N is integrable, there exists a partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ such that $U(f_N, P) - L(f_N, P) < \epsilon/3$. Note that for any interval $[t_{k-1}, t_k]$,

$$\begin{aligned} M(f, [t_{k-1}, t_k]) &\leq M\left(f_N + \frac{\epsilon}{3(b-a)}, [t_{k-1}, t_k]\right) \\ &= M(f_N, [t_{k-1}, t_k]) + \frac{\epsilon}{3(b-a)} \end{aligned}$$

and hence

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &\leq \sum_{k=1}^{n} \left(M(f_N, [t_{k-1}, t_k]) + \frac{\epsilon}{3(b-a)} \right) (t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} M(f_N, [t_{k-1}, t_k])(t_k - t_{k-1}) + \sum_{k=1}^{n} \frac{\epsilon(t_k - t_{k-1})}{3(b-a)} \\ &= U(f_N, P) + \frac{\epsilon(t_n - t_0)}{3(b-a)} \\ &= U(f_N, P) + \frac{\epsilon}{3}. \end{aligned}$$

By similar logic, $L(f, P) \geq L(f_N, P) - \epsilon/3$, so

$$U(f, P) - L(f, P) \leq U(f_N, P) - L(f_N, P) + \frac{2\epsilon}{3} < \epsilon.$$

Since this is true for arbitrary $\epsilon > 0$, it follows that f is integrable, and since the upper and lower sums of f approach the upper and lower sum of the f_N, then $\int_a^b f_n = \int_a^b f$.

(b) Write $u = \log x$ and $dv = 1$. Then $v = x$ and $u = 1/x$, so

$$\int_{1/2}^{1} \log x \, dx = [x \log x]_{1/2}^{1} - \int_{1/2}^{1} dx = -\frac{\log \frac{1}{2}}{2} - \frac{1}{2} = \frac{\log 2}{2} - \frac{1}{2}.$$

(c) Since power series converge uniformly on any open interval $(-c, c)$ where c is smaller than the radius of convergence, then the result from part (a) shows that the sum and integral can be switched, so that

$$\begin{aligned}\int_{1/2}^{1} \log x\, dx = \int_{-1/2}^{0} \log(1+x)\, dx &= \sum_{n=1}^{\infty} \int_{-1/2}^{0} \frac{x^n(-1)^{n+1}}{n} \\ &= \sum_{n=1}^{\infty} \left[\frac{x^{n+1}(-1)^{n+1}}{(n+1)n}\right]_{-1/2}^{0} \\ &= -\sum_{n=1}^{\infty} \frac{1}{2^{n+1}n(n+1)}.\end{aligned}$$

Using part (b),

$$\log 2 = 2\left(\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}n(n+1)}\right) = 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}n(n+1)}.$$