Math 104: Midterm 2 sample solutions

1. To show that f is uniformly continuous, choose $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists an N such that $n > N$ implies that

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

for all $x \in (a,b)$. Now consider f_{N+1}: since this is uniformly continuous, there exists a $\delta > 0$ such that if $x, y \in (a,b)$ and $|x-y| < \delta$, then

$$|f_{N+1}(x) - f_{N+1}(y)| < \frac{\epsilon}{3}.$$

Now, for any $x, y \in (a,b)$ with $|x-y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(y)| + |f_{N+1}(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

and hence f is uniformly continuous.

2. For three values 0, 1, and 2,

$$d_1(0,1) + d_1(1,2) = 1^4 + 1^4 = 2$$

but

$$d_1(0,2) = 2^4 = 16$$

and hence the triangle inequality is violated, so d_1 is not a metric.

Since $d_2(0,0) = 1$, it does not satisfy the property that $d(x,x) = 0$ for all $x \in \mathbb{R}$, and hence d_2 is not a metric. Since $d_3(0,1) = 2$, and $d_3(1,0) = 1$ it is not symmetric, and hence it is not a metric.

3. First, consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n^{\sqrt{n}}}$$

so that the coefficients are $a_n = n^{-\sqrt{n}}$. Then

$$\begin{aligned} \beta &= \limsup |a_n|^{1/n} \\ &= \limsup |n^{-\sqrt{n}}|^{1/n} \\ &= \limsup n^{-1/\sqrt{n}}. \end{aligned}$$

Since the power will always be negative, all of the terms in this sequence must be less than or equal to 1, so $\beta \leq 1$. Consider the subsequence of terms for $n = 2^k$. Then

$$n^{-1/\sqrt{n}} = 2^{-k/2^{k/2}}.$$

It is known that $k/2^{k/2} \to 0$ as $k \to \infty$. Hence $2^{-k/2^{-k}} \to 1$ as $k \to \infty$. Hence, since the subsequence tends to 1 as $n \to \infty$, then $\beta \geq 1$. Combining with the result above, $\beta = 1$, and the radius convergence is $R = 1$.

For the second series

$$\sum_{n=0}^{\infty} 4^n \, x^{2n+1}$$

then

$$a_n = \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Consider the subsequence of odd terms:

$$|a_{2k+1}|^{1/(2k+1)} = |2^{2k+1-1}|^{1/(2k+1)} = 2^{2k/(2k+1)}$$

which converges to 2 as $k \to \infty$. Hence

$$\beta = \limsup |a_n|^{1/n} = 2$$

and therefore the radius of convergence is $R = 1/2$.

For the third series,

$$\sum_{n=0}^{\infty} x^{n^2},$$

the coefficients are $a_n = 1$ if n is a square number and zero otherwise. Hence

$$|a_n|^{1/n} = \begin{cases} 1 & \text{if } n \text{ is a square number} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\beta = \limsup |a_n|^{1/n} = 1$$

since there are infinitely many terms which are 1 in the sequence. Hence the radius of convergence is $R = 1$.

4. If g is bounded on S, then there exists an $M > 0$ such that $|g(x)| < M$ for all $x \in S$. Now consider any $\epsilon > 0$. If f_n converges uniformly to f, then there exists an N such that $n > N$ implies that

$$|f_n(x) - f(x)| < \frac{\epsilon}{M}$$

for all $x \in S$. Now consider $g \cdot f_n$:

$$|g(x)f_n(x) - g(x)f(x)| = |g(x)| \cdot |f_n(x) - f(x)| < M\frac{\epsilon}{M} = \epsilon$$

and hence it uniformly converges to $g \cdot f$.

5. If f_n converges uniformly to f, then there exists an N such that $n > N$ implies that

$$|f_n(x) - f(x)| < 1$$

for all $x \in S$. By using the triangle inequality,

$$|f(x)| < |f_{N+1}(x)| + 1.$$

Since f_{N+1} is bounded, $|f_{N+1}(x)| < M$ for all x and for some $M \geq 0$, and thus $|f(x)| < M + 1$ for all x. Hence f is bounded.

6. Since a continuous function on a closed interval is bounded, then for each f_n, there exists M_n such that $|f_n(x)| < M_n$ for all $x \in [0,1]$. Since f_n converges uniformly to f, there exists an $N \in \mathbb{N}$ such that for all $n > N$,

$$|f_n(x) - f(x)| \leq 1$$

Since the f_n converge uniformly and are continuous, the limit f is continuous also, and therefore bounded, so that $|f(x)| < M'$ for all $x \in [0,1]$, for some $M' > 0$. By using the triangle inequality,

$$|f_n(x)| < |f(x)| + 1 < M' + 1$$

for all $n > N$. Now define

$$L = \max\{M_1, M_2, \ldots, M_N, M' + 1\}$$

Consider any $x \in [0,1]$ and any $n \in \mathbb{N}$. If $n \leq N$, then $|f_n(x)| < M_n \leq L$. If $n > N$, then $|f_n(x)| < M' + 1 \leq L$. Thus L is an upper bound for the set $A = \{|f_n(x)| : n \in \mathbb{N},\ x \in [0,1]\}$. Hence $0 \leq \sup A \leq L$, and thus $\sup A$ must be finite.

7. (a) To show that d is a metric, consider the three properties:

M1. $d(x,x) = \min\{0,1\} = 0$. For any $x, y \in \mathbb{R}$, $d(x,y) = 0$ implies that $|x - y| = 0$, and hence $x = y$. Hence the function is zero if and only if $x = y$.

M2. $d(x,y) = \min\{|x-y|, 1\} = \min\{|y-x|, 1\} = d(y,x)$. Hence the function is symmetric.

M3. Consider any $x, y, z \in \mathbb{R}$. Then

$$d(x,y) + d(y,z) = \min\{|x-y|, 1\} + \min\{|y-z|, 1\}.$$

If either $|x-y| \geq 1$ or $|y-z| \geq 1$, then one of the terms on the right hand side evaluates to 1. Since both terms are positive, then $d(x,y) + d(y,z) \geq 1$. However, $d(x,z) \leq 1$, and thus $d(x,z) \leq d(x,y) + d(y,z)$ so the triangle inequality holds.

Suppose $|x-y| < 1$ and $|y-z| < 1$. Then, by making use of the usual triangle inequality in $\mathbb{R}$,

$$d(x,y) + d(y,z) = |x-y| + |y-z| \leq |(x-y) + (y-z)| = |x-z| \geq d(x,z).$$

Hence d satisfies the triangle inequality.

Hence d is a metric.

(b) Consider any point $x \in (-5, 5)$, and define $r = \min\{1/2, 5 - |x|\}$. Since $|x| < 5$, then $r > 0$. Since $r < 1$, then $N_r(x) = (x - r, x + r)$. Hence

$$x + r \leq x + 5 - |x| \leq 5$$

and

$$x - r \geq x - 5 + |x| \geq -5.$$

Hence $N_r(x) \subseteq (-5, 5)$ so x is an interior point. Since this is true for any $x \in (-5, 5)$, it follows that $(-5, 5)$ is open.

8. (a) For f_1,

$$\begin{aligned} \beta &= \limsup |n^{-2}|^{1/n} \\ &= \limsup \frac{1}{n^{2/n}} = 1 \end{aligned}$$

Hence the radius of convergence is $R = 1$.

For f_2, since some terms are zero, the radius of convergence can be evaluated by computing the limit of the non-zero terms:

$$\begin{aligned} \beta &= \limsup |a_n|^{1/n} \\ &= \lim_{k \to \infty} |a_{2k}|^{1/2k} \\ &= \lim_{k \to \infty} |2^{-k}|^{1/2k} \\ &= \lim_{k \to \infty} 2^{-1/2} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Hence $R = \sqrt{2}$.

(b) Define $x = y/(1 + y^2)$. Then $f_3(y) = f_1(x)$. If $|y| \leq 1$, then $|y| < 1 + y^2$, and hence $|x| < 1$. If $|y| > 1$, then $|y| < y^2$ and so $|y| < 1 + y^2$, so $|x| < 1$ also. Hence for all $y \in \mathbb{R}$, $|x| < 1$, and since $f_1(x)$ converges for x in this range, $f_3(y)$ must converge also.

This question can also be answered using the Weierstraß M-test, by showing that the nth term in the series is bounded by $1/n^2$, and $\sum |1/n^2|$ converges.

9. (a) For $0 \leq x \leq 1$,

$$g(x) = x(1 - x) = x - x^2 = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2.$$

Thus the maximum value is $1/4$, which is attained for $x = 1/2$.

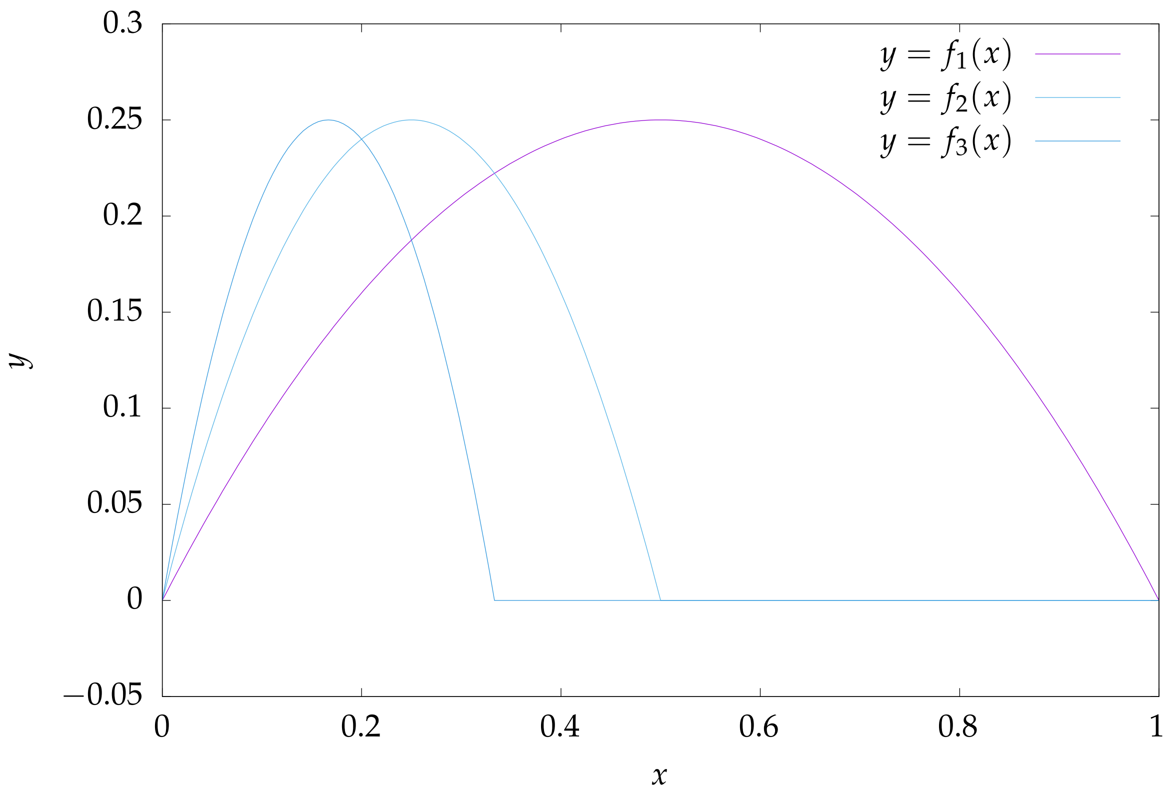

Figure 1: A graph of several of the functions (f_n) for question 3.

(b) The functions are plotted in Fig. 1. A function f_n looks like the function g, but scaled horizontally by a factor of $1/n$. It is important to note that f_n does not become negative, *e.g.*

$$f_2(3/4) = g(2 \times 3/4) = g(3/2) = 0.$$

(c) Since $f_n(0) = 0$ for all n, then $f_n(0) \to 0$. Now choose any $x \in (0, 1]$. Then by the Archimedean property, exists an N such that $1/N < x$. Then for any $n > N$, $f_n(x) = g(nx) = 0$, since $nx > 1$. Hence for any $\epsilon > 0$, then $|f_n(x) - 0| < \epsilon$ for $n > N$, so $f_n(x) \to 0$. Hence f_n converges pointwise to the function $f(x) = 0$.

(d) If f_n converges uniformly, then there exists an N such that $n > N$ implies that $|f_n(x) - f(x)| < 1/4$ for all $x \in [0, 1]$. However, for any n, if $x = 1/(2n)$, then

$$|f_n(x) - f(x)| = |g\left(\frac{n}{2n}\right) - 0| = |g(1/2)| = 1/4.$$

Hence f_n does not converge uniformly to f.

10. To show that $f_n(x_n)$ converges to $f(x)$, consider any $\epsilon > 0$. Since the f_n are continuous and converge uniformly to f, then f must be continuous also. Furthermore, since the interval is closed the limit point x must be within $[a, b]$. Hence, since f is continuous at x, then $f(x_n) \to f(x)$ and hence there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies

$$|f(x_n) - f(x)| < \frac{\epsilon}{2}.$$

In addition, since f_n converges uniformly to f, then there exists an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies

$$|f_n(y) - f(y)| < \frac{\epsilon}{2}$$

for all $y \in [a, b]$. Hence if $N = \max\{N_1, N_2\}$, then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $\lim_{n\to\infty} f_n(x_n) = f(x)$.