Solutions to sample midterm questions

1. Let $x = 1 + \sqrt{1+\sqrt{2}}$. Then

$$\begin{aligned} x-1 &= \sqrt{1+\sqrt{2}} \\ (x-1)^2 &= 1+\sqrt{2} \\ x^2-2x+1 &= 1+\sqrt{2} \\ x^2-2x &= \sqrt{2} \\ x^4-4x^3+4x^2 &= 2 \\ x^4-4x^3+4x^2-2 &= 0 \end{aligned}$$

Hence if $x = p/q$, then p divides 2 and q divides 1. The only possibilities are ± 1, and ± 2. But $\sqrt{1+\sqrt{2}} > 1$, and thus $x > 2$. Thus x must be irrational.

2. Define $a_n = 8^n/(n!)^2$. Then

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{8^{n+1}}{((n+1)!)^2}\frac{(n!)^2}{8^n} \\ &= \frac{8}{(n+1)^2} \to 0 \end{aligned}$$

and therefore by the ratio test, $\sum 8^n/(n!)^2$ converges.

Now consider $\sum(-1)^n b_n$ where $b_n = 1/\sqrt{n^2+n}$. Since n and n^2 are both increasing functions, n^2+n is an increasing function also, and hence $1/\sqrt{n^2+n}$ is a decreasing function. In addition, $b_n < 1/n$ for all n, so $b_n \to 0$ as $n \to \infty$. Therefore, the series $\sum(-1)^n b_n$ satisfies the conditions for the alternating series theorem, and hence it converges.

3. (a) Let $x \in S \cup T$. Then either $x \in S$ so $x \leq \sup S$, or $x \in T$ so $x \leq \sup T$. Hence, $x \leq \max\{\sup S, \sup T\}$. Thus $\max\{\sup S, \sup T\}$ is an upper bound for $\sup S \cup T$.

   Now suppose that m is an upper bound for $S \cup T$. Hence $m \geq x$ for all $x \in S \cup T$. Thus $m \geq s$ for all $s \in S$, so $m \geq \sup S$ as $\sup S$ is the least upper bound for S. Similarly $m \geq t$ for all $t \in T$. Hence $m \geq \sup T$ as $\sup T$ is the least upper bound of T. Therefore $m \geq \max\{\sup S, \sup T\}$. Hence $\max\{\sup S, \sup T\}$ is an upper bound, and it is the least upper bound, so it must equal $\sup S \cup T$.

   Now consider $x \in S \cap T$. Hence $x \in S$ and $x \in T$. Then $x \leq \sup S$ and $x \leq \sup T$, so $x \leq \min\{\sup S, \sup T\}$, and therefore $\sup S \cap T \leq \min\{\sup S, \sup T\}$.

   (b) For a non-empty set A, $\sup A \neq -\infty$, so it suffices to consider when the suprema become positive infinity. Suppose $\sup S = \infty$. Then S is not bounded above. Hence $S \cup T$ is not bounded above. Therefore $\sup S \cup T = \infty$ and the identity still holds.

For the second identity, if $\sup S = \infty$, then $\min\{\sup S, \sup T\} = \sup T$. Since $\sup T$ is an upper bound for T, it is also an upper bound for $S \cap T$, and hence the identity still holds.

The same arguments can be applied if $\sup T = \infty$.

(c) Consider $S = \{1,3\}$ and $T = \{1,2\}$. Then $\sup S = 3$ and $\sup T = 2$, so $\min\{\sup S, \sup T\} = 2$. However, $S \cap T = \{1\}$ and so $\sup S \cap T = 1 < 2$.

4. Let $\lim s_n = s$. Since s_n converges, there exists an N_1 such that $n > N_1$ implies that $|s_n - s| < 1$. Hence $-1 < s_n - s$ and $s_n > s - 1$.

   Now pick $M > 0$. Since t_n diverges, there exists an N_2 such that

$$t_n > 1 - s + M$$

   for all $n > N_2$. Hence for $n > \max\{N_1, N_2\}$,

$$s_n + t_n > (s-1) + 1 - s + M = M$$

   and thus $s_n + t_n$ diverges to infinity.

5. (a) Define $a_N = \sup\{s_n : n > N\}$ and $b_N = \sup\{t_n : n > N\}$. Now, for $n > N$,

$$s_n + t_n \le a_N + b_N$$

   since a_N and b_N are upper bounds for s_n and t_n. If $c_N = \sup\{s_n + t_n : n > N\}$, then

$$c_N \le a_N + b_N.$$

   The sequences (a_N), (b_N), and (c_N) are non-increasing. Suppose that $\lim c_N > \lim a_N + \lim b_N$. Then $\lim c_N = \lim a_N + \lim b_N + \epsilon$ for some $\epsilon > 0$, so there exist K_1 and K_2 such that if $k > K_1$

$$a_k < \lim a_N + \frac{\epsilon}{3}$$

   and if $k > K_2$ then

$$b_k < \lim b_N + \frac{\epsilon}{3}.$$

   Now for all $k > \max\{K_1, K_2\}$,

$$\begin{aligned} c_k &\le a_k + b_k \\ &< \left(\lim a_N + \frac{\epsilon}{3}\right) + \left(\lim b_N + \frac{\epsilon}{3}\right) \\ &< (\lim a_N + \lim b_N + \epsilon) - \frac{\epsilon}{3} = \lim c_N - \frac{\epsilon}{3}. \end{aligned}$$

   But then $|c_k - \lim c_N| > \frac{\epsilon}{3}$ for all $k > \max\{K_1, K_2\}$, so c_k does not converge to $\lim c_N$ which is a contradiction. Hence $\lim c_N \le \lim a_N + \lim b_N$, and hence $\limsup s_n + t_n \le \limsup s_n + \limsup t_n$.

(b) Suppose

$$s_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

and that

$$t_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Then $\sup\{s_n : n > N\}$ and $\sup\{t_n : n > N\} = 1$ for all $N \in \mathbb{N}$, and hence

$$(\limsup s_n) \cdot (\limsup t_n) = 1 \cdot 1 = 1.$$

However, $s_n t_n = 0$ for all n, and thus $\limsup(s_n t_n) = 0 \neq 1$.

6. Suppose that $a > b$. Then define $\epsilon = a - b > 0$. Then there exists an N_1 such that $n > N_1$ implies that $|a_n - a| < \epsilon/2$. Similarly there exists an N_2 such that $n > N_2$ implies that $|b_n - b| < \epsilon/2$. Now consider any k such that $k > \max\{N_1, N_2\}$. Then $|a_k - a| < \epsilon/2$, and hence $-\epsilon/2 < a_k - a$, so

$$a_k > a - \frac{\epsilon}{2} = a - \frac{a-b}{2} = \frac{a+b}{2}.$$

In addition, $|b_k - b| < \epsilon/2$, so $b_k - b < \epsilon/2$, and hence

$$b_k < b + \frac{\epsilon}{2} = b + \frac{a-b}{2} = \frac{a+b}{2}.$$

Combining these two inequalities shows that $a_k > b_k$, which is a contradiction. Thus $a \leq b$.

7. Since the lower limit of A is an open interval, it does not have a minimum, however $\inf A = 0$. Since A is not bounded above, it does not have a maximum. $\sup A = \infty$ for sets not bounded above.

Since B has no smallest element, the minimum does not exist. However, since the fractions become arbitrarily close to 0, $\inf B = 0$. The maximum is given by $\max B = 1/2$, attained for the case when $n = 1$, and hence $\sup B = \max B = 1/2$.

8. For the first sequence, make use of the root test where $a_n = 6^n/n^n$. Then

$$(a_n)^{1/n} = \frac{6}{n}$$

which converges to zero as $n \to \infty$. Hence $\sum 6^n/n^n$ converges. For the second sequence, since $n + 1/2 \leq 2n$ for all $n \in \mathbb{N}$, then

$$\frac{1}{n + 1/2} \geq \frac{1}{2n}$$

for all $n \in \mathbb{N}$. Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{2n}$, and hence by the comparison test, $\sum 1/(n + 1/2)$ does also.

9. Choose an element $t \in T$. Then either

   - $t \in S$. Hence $t \leq \sup S$.
   - There exists $s \in S$ such that $s = -t$. Hence $s \geq \inf S$, and therefore $t \leq -\inf S$.

   Thus either $t \leq \sup S$ or $t \leq -\inf S$ so $t \leq \max\{\sup S, -\inf S\}$. Hence it is an upper bound.

   Now suppose that l is an upper bound for T. Then $l \geq t$ for all elements $t \in T$. Hence $l \geq |s|$ for all elements $s \in S$, and thus

   $$-l \leq s \leq l$$

   for all elements in s, from which the following two deductions can be made:

   - Since $s \leq l$ for all s, then $l \geq \sup S$ since $\sup S$ is the least upper bound for S.
   - Since $-l \leq s$ for all s, then $-l \leq \inf S$ since $\inf S$ is the greatest lower bound for S. Hence $l \geq -\inf S$.

   These two results show that $l \geq \max\{\sup S, -\inf S\}$. Hence $\max\{\sup S, -\inf S\}$ is an upper bound for T and it is the least upper bound, so it must be $\sup T$.

10. Define $v_N = \sup\{s_n : n > N\}$. There are two cases:

    - $\limsup t_n = -q$ for some $q > 0$. Then there exists a K_1 such that $|v_N - (-q)| < q/2$ for all $N > K_1$. Hence $v_{K_1+1} < (-q) + (q/2) = -q/2$, and thus $t_n < -q/2$ for all $n > K_1 + 1$. For this case, define $\lambda = -q/2$.
    - $\limsup t_n = -\infty$. Then there exists a K_1 such that $v_N < -1$ for all $N > K_1$. Hence $v_{K_1+1} < -1$, and thus $t_n < -1$ for all $n > K_1 + 1$. For this case, define $\lambda = -1$.

    Now consider the sequence $s_n t_n$. Pick $M < 0$. Then since $\lim s_n = \infty$, there exists a K_2 such that $n > K_2$ implies that $s_n > M/\lambda$.

    Now suppose $n > \max\{K_1 + 1, K_2\}$. Then $s_n > M/\lambda$ and $t_n < \lambda$, so $s_n t_n < M$. This is true for any $M < 0$, so $\lim s_n t_n = -\infty$.