Math 104: Midterm 2 information

- The second class midterm will take place in class on March 21st from 4:10pm–5pm.
- The exam will cover everything in class from metric spaces up to the lectures on power series on Friday 17th. This corresponds to the first half of chapter 13 in Ross stopping just before Definition 13.11, and then everything in chapters 17–20 and 23–27.
- The exam is closed book – no textbooks, notebooks, or calculators allowed. As on the first midterm, you will not be expected to know every theorem by heart, but you will be expected to remember the definitions, such as metrics, open/closed sets, interiors and closures, uniform continuity, uniform convergence, and the radius of convergence for power series. You will also need to remember some of the key results, such as the Intermediate Value Theorem, and the theorem that a continuous function on a closed interval is bounded and achieves its bounds.
- There will not be any questions that specifically cover the material from the first midterm (*eg.* suprema, sequences, series). However, since many of the more recent topics build on this prior work, you will need to be familiar with this.
- There will be two short questions, followed by two longer questions involving mathematical proof.

Sample midterm questions

Questions 7–10 were used in the second midterm when the class was taught in Spring 2011.

1. Let (f_n) be a sequence of uniformly continuous functions on an interval (a, b), and suppose that f_n converges uniformly to a function f. Prove that f is uniformly continous on (a, b).

2. Prove that the functions

$$d_1(x, y) = (x - y)^4, \qquad d_2(x, y) = 1 + |x - y|,$$

$$d_3(x, y) = \begin{cases} x - y & \text{if } x > y \\ 2(y - x) & \text{if } x \leq y \end{cases}$$

are not metrics on $\mathbb{R}$.

3. Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n\sqrt{n}}, \qquad \sum_{n=0}^{\infty} 4^n x^{2n+1}, \qquad \sum_{n=0}^{\infty} x^{n^2}.$$

4. Suppose that f_n converges uniformly to f on a set $S \subseteq \mathbb{R}$, and that g is a bounded function on S. Prove that the multiplication $g \cdot f_n$ converges uniformly to $g \cdot f$.

5. Let (f_n) be a sequence of bounded functions on a set S, and suppose that $f_n \to f$ uniformly on S. Prove that f is a bounded function on S.

6. Let (f_n) be a sequence of real-valued continuous functions defined on the interval $[0,1]$. Suppose that f_n converges uniformly to a function f. Define a global bound M according to
$$M = \sup\{|f_n(x)| \,:\, n \in \mathbb{N},\, x \in [0,1]\}.$$
Prove that M is finite.

7. (a) Prove that the function
$$d(x,y) = \min\{|x-y|, 1\}$$
is a metric on $\mathbb{R}$.

   (b) Is the set $(-5,5)$ open with respect to this metric? Prove your assertion.

8. (a) Find the radius of convergence of the power series
$$f_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad f_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}.$$

   (b) Show that the series
$$f_3(y) = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{y}{1+y^2}\right)^n$$
converges for all values of $y \in \mathbb{R}$.

9. Consider the function defined on the domain $[0,\infty)$ as
$$g(x) = \begin{cases} x(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{if } x > 1. \end{cases}$$
Define a sequence of functions on the interval $[0,1]$ according to $f_n(x) = g(nx)$.

   (a) What is the maximum value of $g(x)$, and where is it attained?

   (b) Sketch the functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ on $[0,1]$.

   (c) Prove that f_n converges pointwise to a function f on $[0,1]$, and determine f.

   (d) Does f_n converge uniformly to f on $[0,1]$? Prove your assertion.

10. Let (f_n) be a sequence of continuous functions on $[a,b]$ that converges uniformly to f on $[a,b]$. Show that if (x_n) is a sequence in $[a,b]$ and if $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$.