Math 104: Midterm information

- The first class midterm will take place in class on Friday February 24th from 4:10pm–5pm.
- The exam will cover everything in class up to and including alternating series. This corresponds to chapters 1–5, 7–12, 14, and 15 in Ross. The more recent material on metric spaces will not be covered.
- The exam is closed book – no textbooks, notebooks, or calculators allowed. You will not be expected to know every theorem by heart, but you will be expected to remember the basic definitions, such as suprema/infima, convergence/divergence, lim sup, lim inf, and subsequential limits. You should also be familiar with the limit theorems for determining convergence of sequences, and the tests for determining series convergence and divergence.
- There will be two short questions, followed by two longer questions which will involve constructing a mathematical proof.
- If a question uses the word "prove", then you will be expected to write down a mathematical proof similar to those in the textbook or given in the homework solutions. If a question uses weaker langauge, such as "determine", "justify", or "compute", a less rigorous argument will still receive full credit.
- You can assume a number of basic results, such as
  - ◇ $\sum n^{-p}$ converges if and only if $p > 1$,
  - ◇ the triangle inequality: $|a| + |b| \geq |a + b|$ for all $a, b \in \mathbb{R}$,
  - ◇ $\lim_{n\to\infty} a^n = 0$ for $|a| < 1$,
  - ◇ $|b| < a$ if and only if $-a < b < a$.

Sample midterm questions

The following questions are of a similar style to the ones that will be on the midterm. They are designed to test familiarity with basic concepts, and will generally be more straightforward than some of the questions on the homework. Questions 7–10 were used in the first midterm when the class was taught in 2011.

1. Prove that $1 + \sqrt{1 + \sqrt{2}}$ is irrational.
2. Consider the following series defined for $n \in \mathbb{N}$:

$$\sum \frac{8^n}{(n!)^2}, \qquad \sum \frac{(-1)^n}{\sqrt{n^2 + n}}.$$

For each series, determine whether they converge or diverge. If you make use of any of the theorems for determining series properties, you should state which one you use.

3. (a) Let S and T be non-empty bounded subsets of $\mathbb{R}$. Prove that $\sup S \cup T = \max\{\sup S, \sup T\}$ and $\sup S \cap T \leq \min\{\sup S, \sup T\}$.
   (b) Extend part (a) to the cases where S and T are not bounded.
   (c) Give an example where $\sup S \cap T < \min\{\sup S, \sup T\}$.

4. Suppose that (s_n) is a convergent sequence and (t_n) is a sequence that diverges to ∞. Prove that
$$\lim_{n\to\infty} s_n + t_n = \infty.$$

5. (a) Let (s_n) and (t_n) be two sequences defined for $n \in \mathbb{N}$. Prove that $\limsup s_n + \limsup t_n \geq \limsup(s_n + t_n)$.
   (b) Construct an example where $(\limsup s_n) \cdot (\limsup t_n) \neq \limsup(s_n t_n)$.

6. Let (a_n) and (b_n) are sequences defined for $n \in \mathbb{N}$. Suppose that $a_n \to a$ and $b_n \to b$ as $n \to \infty$ for some $a, b \in \mathbb{R}$. If $a_n \leq b_n$ for all n, show that $a \leq b$.

7. Consider the two sets
$$A = (0, 1] \cup [4, \infty), \qquad B = \left\{ \frac{1}{2n} \,:\, n \in \mathbb{N} \right\}.$$
For each set, determine its maximum and minimum if they exist. For each set, determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

8. Consider the following series, defined for $n \in \mathbb{N}$:
$$\sum \frac{6^n}{n^n}, \qquad \sum \frac{1}{n + 1/2}.$$
For each series, determine whether it converges or diverges. If you make use of any of the theorems for determining series properties, you should state which one you use.

9. Let S be a non-empty bounded subset of $\mathbb{R}$. Define $T = \{|x| \,:\, x \in S\}$ to be the set of all absolute values of elements in S. Prove that $\sup T = \max\{\sup S, -\inf S\}$.

10. Let (s_n) and (t_n) be two sequences defined for $n \in \mathbb{N}$. Suppose $\lim s_n = \infty$, and $\limsup t_n < 0$. Prove that $\lim s_n t_n = -\infty$.

    Note: make sure to consider both cases when $\limsup t_n$ *is a real number, and when* $\limsup t_n$ *is* $-\infty$.