# Math 104: Midterm 2 solutions

1. Consider the power series
$$\sum_{n=1}^{\infty} \frac{(2x)^{n^2}}{n}.$$

   (a) Show that the series diverges at $x = 1/2$ and converges at $x = -1/2$.

   **Answer:** If $x = 1/2$, the series becomes
$$\sum_{n=1}^{\infty} \frac{1}{n},$$
   which diverges to $\infty$. If $x = -1/2$, the series becomes
$$\sum_{n=1}^{\infty} \frac{(-1)^{n^2}}{n}.$$
   If $n$ is even then $n^2$ is even. If $n$ is odd, then it can be written as $2k+1$, and thus $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so $n^2$ is odd. Hence $(-1)^{n^2} = (-1)^n$ for all $n \in \mathbb{N}$ and thus the series can be rewritten as
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$
   This converges by the alternating series theorem.

   (b) What is the radius of convergence of the series? Either calculate it explicitly, or justify carefully using part (a).

   **Answer:** If the radius of convergence of a power series is $R$, then it must converge for $|x| < R$ and diverge for $|x| > R$. Since the series converges for $x = 1/2$, it follows that $R \geq 1/2$. Since the series diverges for $x = -1/2$, it follows that $R \leq 1/2$. Hence $R = 1/2$.

   Alternatively, to calculate explicitly, first rewrite the sum as $\sum a_n x^n$, where $a_n = 2^n n^{-1/2}$ if $n$ is a square number, and $a_n = 0$ otherwise. The general radius of convergence formula then gives
$$\beta = \limsup |a_n|^{1/n} = \limsup |a_{n^2}|^{1/n^2} = \lim_{n\to\infty} \left| \frac{2^{n^2}}{n} \right|^{1/n^2} = 2 \lim_{n\to\infty} |n|^{-1/n^2}.$$
   For all $n \in \mathbb{N}$, $|n|^{-1/n} \leq |n|^{-1/n^2} \leq 1$. Since $\lim_{n\to\infty} |n|^{-1/n} = 1$, it follows that $\lim_{n\to\infty} |n|^{-1/n^2} = 1$ by the squeezing lemma. Hence $\beta = 2$, so $R = 1/\beta = 1/2$.

2. Consider the function

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 + |x-y| & \text{if } x \neq y, \end{cases}$$

defined for all $x, y \in \mathbb{R}$.

(a) Prove that $d$ is a metric on $\mathbb{R}$.

**Answer:** Consider the three properties of a metric:

M1. From the definition, it can be seen that $d(x,x) = 0$ for all $x \in \mathbb{R}$, and that $d(x,y) > 0$ for all $x, y \in \mathbb{R}$ where $x \neq y$.

M2. Consider $x, y \in \mathbb{R}$. If $x = y$ then $d(x,y) = 0 = d(y,x)$. If $x \neq y$, then $d(x,y) = 1 + |x-y| = 1 + |y-x| = d(y,x)$ and hence $d$ is symmetric.

M3. Consider $x, y, z \in \mathbb{R}$. If $x = y$, then $d(x,y) + d(y,z) = 0 + d(y,z) = d(x,z)$ and the triangle inequality is satisfied. If $y = z$, then similarly $d(x,y) + d(y,z) = d(x,z)$. Otherwise $x \neq y$ and $y \neq z$. By making use of the usual triangle inequality,

$$d(x,y) + d(y,z) = 2 + |x-y| + |y-z| \geq 2 + |x-z| > 1 + |x-z| \geq d(x,z).$$

Hence $d$ satisfies the triangle inequality.

(b) Find the interior of $[0,1]$ with respect to $d$.

**Answer:** For any $x \in [0,1]$, the neighborhood of radius $1/2$ at $x$ is $N_{1/2}(x) = \{y \in \mathbb{R} : d(x,y) < 1/2\} = \{x\}$. Since $\{x\} \subseteq [0,1]$, it follows that $x$ is an interior point. Hence the interior is $[0,1]$.

(c) Suppose that $(s_n)$ is a Cauchy sequence in $\mathbb{R}$ with respect to $d$. Prove that it is a convergent sequence with respect to $d$.

**Answer:** Since $(s_n)$ is Cauchy, there exists an $N$ such that $n, m > N$ implies that $d(s_n, s_m) < 1/2$. However, $d(x,y) > 1$ for any $x \neq y$. Hence $s_n = s_m$ for all $n, m > N$, so the sequence is equal to some constant $s$ for all $n > N$. Hence, for any $\epsilon > 0$, $n > N$ implies that $d(s_n, s) = 0 < \epsilon$, so the series converges.

3. Consider the functions $f(x) = x^2(2-x)$ and $g(x) = |f(x)|$ defined for all $x \in \mathbb{R}$.

(a) Sketch $f$ and $g$ over the domain $-1 \leq x \leq 3$.

**Answer:** The functions are sketched in Figure 1.

(b) Use the $\epsilon$-$\delta$ property to prove that $g$ is continuous at $x = 2$.

**Answer:** Consider any $\epsilon > 0$. Then

$$|g(x) - g(2)| = \left| |x^2(2-x)| - 0 \right| = x^2|2-x|.$$

Suppose $|x-2| < 1$. Then $1 < x < 3$, and hence $x^2|2-x| < 3^2|2-x| = 9|2-x|$. If $|x-2| < \delta$ where $\delta = \min\{1, \epsilon/9\}$, then

$$|g(x) - g(2)| < 9|2-x| < \frac{9\epsilon}{9} = \epsilon$$

and hence $g$ is continuous at $x = 2$.

(c) Prove that there are at least four solutions to the equation $g(x) = 1/2$.

**Answer:** By looking at Figure 1, it can be seen that $g(x)$ crosses the line $y = 1/2$ four times. Note that

$$g(-1) = 3, \qquad g(0) = 0, \qquad g(1) = 1, \qquad g(2) = 0, \qquad g(3) = 9$$

and hence applying the Intermediate Value Theorem to the intervals $(-1, 0)$, $(0, 1)$, $(1, 2)$, and $(2, 3)$ will give four distinct values of $x$ where $g(x) = 1/2$.

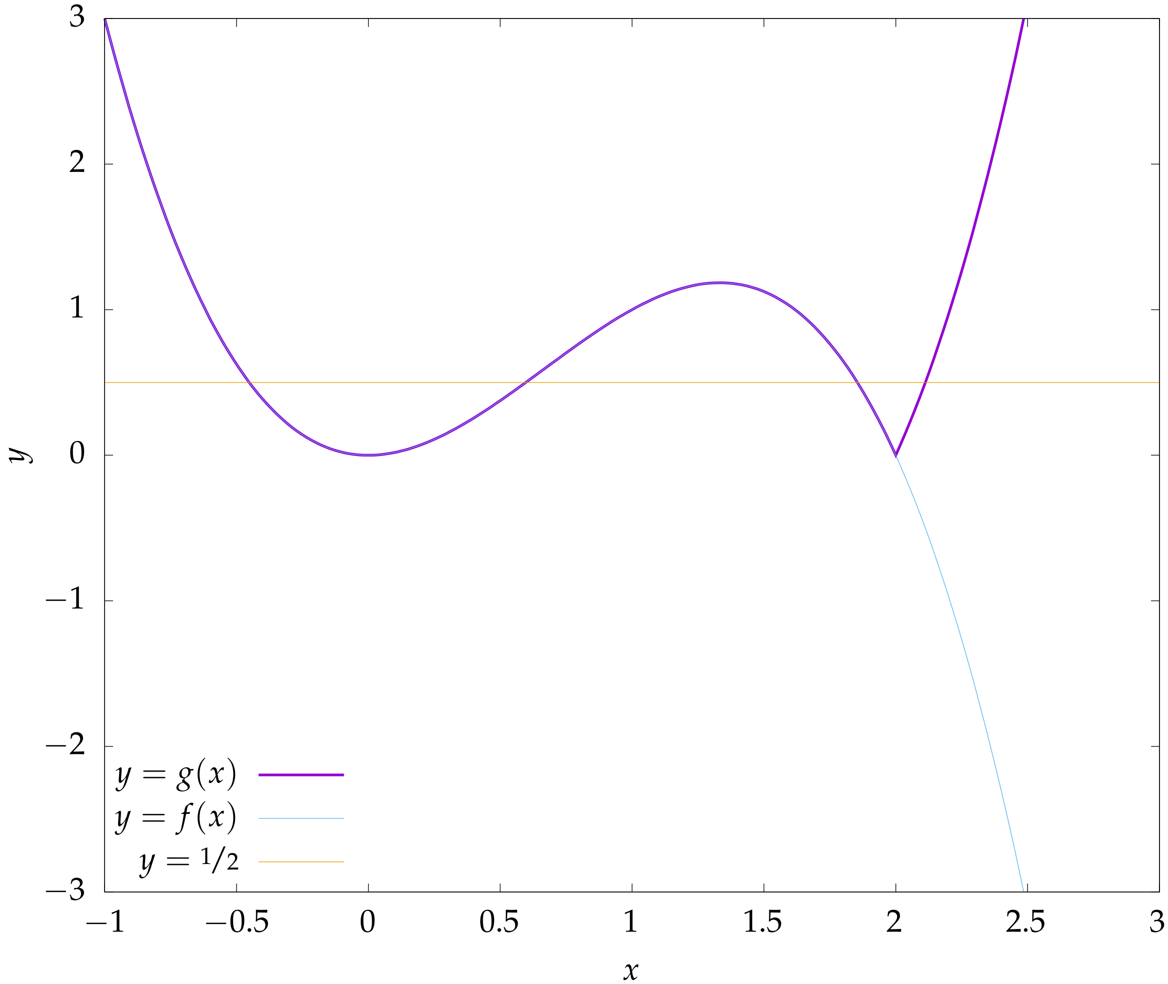


Figure 1: Functions considered in question 3.

4. Let $f$ be a real-valued function on $(0,1)$. Define a sequence of functions as

$$f_n(x) = \begin{cases} \alpha & \text{if } x < 1/n, \\ f(x) & \text{if } x \geq 1/n \end{cases}$$

where $\alpha$ is a real constant.

(a) Prove that $f_n \to f$ pointwise.

**Answer:** Consider a fixed $x \in (0,1)$. Then by the Archimedean principle there exists an $N \in \mathbb{N}$ such that $1/N < x$. Consider any $\epsilon > 0$. Then $n > N$ implies that $|f_n(x) - f(x)| = |f(x) - f(x)| = 0 < \epsilon$, and thus $\lim_{n\to\infty} f_n(x) = f(x)$. Hence $f_n \to f$ pointwise.

(b) Prove that $f_n \to f$ uniformly if and only if $\lim_{x\to 0^+} f(x) = \alpha$.

**Answer:** Suppose $\lim_{x\to 0^+} f = \alpha$. To prove $f_n \to f$ uniformly, consider any $\epsilon > 0$. Then there exists $\delta > 0$ such that $0 < x < \delta$ implies that $|f(x) - \alpha| < \epsilon$. By the Archimedean principle, there exists an $N \in \mathbb{N}$ such that $1/N < \delta$. Consider $f_n$ for $n > N$. If $x \geq 1/n$, then $f_n(x) = f(x)$. If $x < \frac{1}{n}$ then $f_n(x) = \alpha$, and since $x < \delta$, it follows that $|f(x) - \alpha| < \epsilon$, so $|f(x) - f_n(x)| < \epsilon$. Hence $|f(x) - f_n(x)| < \epsilon$ for all $x \in (0,1)$. Hence $f_n \to f$ uniformly.

Now consider the converse and suppose $f_n \to f$ uniformly. For any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in (0,1)$. If $\delta = \frac{1}{N+1}$ then $0 < x < \delta$ implies that $f_{N+1}(x) = \alpha$, and hence $|f(x) - \alpha| = |f(x) - f_{N+1}(x)| < \epsilon$. Hence $\lim_{x\to 0^+} f(x) = \alpha$.