Math 104: Midterm 1 solutions

1. (a) Use the rational zeroes theorem to prove that

$$x = \sqrt{2+\sqrt{3}}$$

is irrational.

Answer: A polynomial for x can be derived as

$$\begin{aligned} x^2 &= 2+\sqrt{3} \\ (x^2-2)^2 &= 3 \\ x^4-4x^2+1 &= 0. \end{aligned}$$

By the rational zeroes theorem, if $x = p/q$ where p and q integers, then p divides 1 and q divides 1. Thus the only possibilities for x are ± 1. It is clear that x is positive, so $x = -1$ is not possible. Since $(1)^4 - 4(1)^2 + 1 = -2$ it follows that $x = 1$ is also not possible. Thus x must be irrational.

(b) Consider the set $T = \{t \in \mathbb{Q} : 0 \leq t \leq x\}$ where x is defined as above. Determine its maximum and minimum if they exist. Determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

Answer: Since $0 \in T$ and $0 \geq t$ for all $t \in T$, it follows that $\min T = 0$. Consider any number $t \in T$. Since $t \in \mathbb{Q}$ then $t < x$. By the denseness of $\mathbb{Q}$, there is a rational number s such that $t < s < x$, and hence t is not a maximum element for T. Thus the maximum does not exist.

Since the minimum is 0, the infimum is equal to 0 also. The supremum is x, since this is an upper bound, and for any value $y < x$ there exists an element in T that is larger than y, so it is the least upper bound.

2. (a) State the definition for a sequence (s_n) to converge to a limit s as $n \to \infty$.

Answer: The sequence (s_n) converges to s if for all $\epsilon > 0$, there exists an N such that for all $n > N$, $|s_n - s| < \epsilon$.

(b) Consider the sequence (s_n) defined for $n \in \mathbb{N}$ as

$$s_n = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Show that (s_n) does not converge.

Answer: Suppose (s_n) converges to a limit s. Consider $\epsilon = 1/2$; there exists N such that $n > N$ implies that $|s_n - s| < 1/2$. Let j be an even number larger than N, and let k be an odd number larger than N. Then

$$|s_j - s_k| = |(s_j - s) - (s_k - s)| \leq |s_j - s| + |s_k - s| < 1/2 + 1/2 = 1.$$

However $|s_j - s_k| = |2 - 3| = 1$, and thus $1 < 1$ which is a contradiction. Thus (s_n) does not converge.

(c) Determine whether the series

$$\sum \frac{1}{(s_n)^n}$$

converges or diverges.

Answer: Note that

$$\left| \frac{1}{(s_n)^n} \right| \leq 2^{-n}.$$

The series $\sum 2^{-n}$ is a geometric series and converges. Hence, by the comparison test, the given series must converge also. Alternatively, by using the root test,

$$\limsup \left| \frac{1}{(s_n)^n} \right|^{1/n} = \limsup \frac{1}{s_n} = \frac{1}{2}$$

and since this is less than 1 the series converges.

3. Let S an T be two non-empty subsets of $(0, \infty)$. Define $R = \{st : s \in S, t \in T\}$ to be the set of all products of elements from S and T.

(a) Suppose that S and T are bounded. Prove that

$$\sup R = (\sup S)(\sup T).$$

Answer: Write $s_0 = \sup S$ and $t_0 = \sup T$. Both s_0 and t_0 are strictly positive since both S and T contain a strictly positive element. Consider an element $r \in R$. Then $r = st$ for some $s \in S$ and $t \in T$. Since $s \leq s_0$ and $t \leq t_0$, it follows that

$$st \leq s_0 t_0,$$

and thus $s_0 t_0$ is an upper bound for R. Now suppose that $q = s_0 t_0 - \epsilon$ is an upper bound for R. It must be the case that $q > 0$, since S and T are sets of positive numbers. Since s_0 is the least upper bound for S, there exists an $s \in S$ such that

$$s > s_0 - \frac{\epsilon}{2t_0}$$

and since t_0 is the least upper bound for T, there exists a $t \in T$ such that

$$t > t_0 - \frac{\epsilon}{2s_0}.$$

Thus

$$st > \left(s_0 - \frac{\epsilon}{2t_0}\right)\left(t_0 - \frac{\epsilon}{2s_0}\right) = s_0 t_0 - \epsilon + \frac{\epsilon}{4s_0 t_0} > s_0 t_0 - \epsilon = q$$

and hence q is not an upper bound of T. Thus $\sup R = s_0 t_0$.

(b) Prove that if $\sup S = \infty$ then $\sup R = \infty$.

Answer: Suppose $\sup S = \infty$. Choose any element $t \in T$. Suppose that M was an upper bound for R. Since $\sup S = \infty$, there exists an $s \in S$ such that $s > M/t$. Hence $st > tM/t = M$. But $st \in R$, which is a contradiction, and thus $\sup R = \infty$.

4. Let (s_n) and (t_n) be sequences such that $\lim s_n = s$ and $\limsup t_n = t$, where s and t are real numbers. Prove that $\limsup(s_n + t_n) = s + t$.

Answer: Consider $\epsilon > 0$. Then there exists an N_1 such that $n > N_1$ implies that

$$|s_n - s| < \frac{\epsilon}{2}$$

and hence

$$s - \frac{\epsilon}{2} < s_n < s + \frac{\epsilon}{2}.$$

Similarly, there exists an N_2 such that $N > N_2$ implies that

$$|\sup\{t_n : n > N\} - t| < \frac{\epsilon}{2}$$

and hence

$$t - \frac{\epsilon}{2} < \sup\{t_n : n > N\} < t + \frac{\epsilon}{2}.$$

Then for $N > \max\{N_1, N_2\}$,

$$\begin{aligned} \sup\{s_n + t_n : n > N\} &\leq \sup\{s + \epsilon/2 + t_n : n > N\} \\ &= s + \frac{\epsilon}{2} + \sup\{t_n : n > N\} \\ &< s + \frac{\epsilon}{2} + t + \frac{\epsilon}{2} = s + t + \epsilon. \end{aligned} \tag{1}$$

The first inequality is obtained because each element in the set is replaced with something larger, so the supremum must be larger. For the second line, the quantity $s + \epsilon/2$ can be brought outside the supremum since it shifts all the terms by a constant amount. By following similar steps,

$$\begin{aligned} \sup\{s_n + t_n : n > N\} &\geq \sup\{s - \epsilon/2 + t_n : n > N\} \\ &= s - \frac{\epsilon}{2} + \sup\{t_n : n > N\} \\ &> s - \frac{\epsilon}{2} + t - \frac{\epsilon}{2} = s + t - \epsilon. \end{aligned} \tag{2}$$

Hence, by making use of Eqs. 1 and 2,

$$|\sup\{s_n + t_n : n > N\} - s - t| < \epsilon$$

for all $N > \max\{N_1, N_2\}$. Since argument can be applied for arbitrary $\epsilon > 0$, it follows that $\limsup(s_n + t_n) = s + t$.