Math 104: Midterm 2

1. Consider the power series
$$\sum_{n=1}^{\infty} \frac{(2x)^{n^2}}{n}.$$
   (a) Show that the series diverges at $x = 1/2$ and converges at $x = -1/2$.
   (b) What is the radius of convergence of the series? Either calculate it explicitly, or justify carefully using part (a).

2. Consider the function
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 + |x-y| & \text{if } x \neq y, \end{cases}$$
defined for all $x, y \in \mathbb{R}$.
   (a) Prove that d is a metric on $\mathbb{R}$.
   (b) Find the interior of $[0,1]$ with respect to d.
   (c) Suppose that (s_n) is a Cauchy sequence in $\mathbb{R}$ with respect to d. Prove that it is a convergent sequence with respect to d.

3. Consider the functions $f(x) = x^2(2-x)$ and $g(x) = |f(x)|$ defined for all $x \in \mathbb{R}$.
   (a) Sketch f and g over the domain $-1 \leq x \leq 3$.
   (b) Use the ϵ-δ property to prove that g is continuous at $x = 2$.
   (c) Prove that there are at least four solutions to the equation $g(x) = 1/2$.

4. Let f be a real-valued function on $(0,1)$. Define a sequence of functions as
$$f_n(x) = \begin{cases} \alpha & \text{if } x < 1/n, \\ f(x) & \text{if } x \geq 1/n \end{cases}$$
where α is a real constant.
   (a) Prove that $f_n \to f$ pointwise.
   (b) Prove that $f_n \to f$ uniformly if and only if $\lim_{x \to 0^+} f(x) = \alpha$.