Math 104: Midterm 1

1. (a) Use the rational zeroes theorem to prove that

$$x = \sqrt{2 + \sqrt{3}}$$

is irrational.

(b) Consider the set $T = \{y \in \mathbb{Q} : 0 \leq y \leq x\}$ where x is defined as above. Determine its maximum and minimum if they exist. Determine its supremum and infimum. Detailed proofs are not required, but you should justify your answers.

2. (a) State the definition for a sequence (s_n) to converge to a limit s as $n \to \infty$.

(b) Consider the sequence (s_n) defined for $n \in \mathbb{N}$ as

$$s_n = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Show that (s_n) does not converge.

(c) Determine whether the series

$$\sum \frac{1}{(s_n)^n}$$

converges or diverges.

3. Let S and T be two non-empty subsets of $(0, \infty)$. Define $R = \{st : s \in S, t \in T\}$ to be the set of all products of elements from S and T.

(a) Suppose that S and T are bounded. Prove that

$$\sup R = (\sup S)(\sup T).$$

(b) Prove that if $\sup S = \infty$ then $\sup R = \infty$.

4. Let (s_n) and (t_n) be sequences such that $\lim s_n = s$ and $\limsup t_n = t$, where s and t are real numbers. Prove that $\limsup(s_n + t_n) = s + t$.