A discussion about taking limits

Overview of the problem

On question 6 of homework 2 the aim was to show that $\lim_{n\to\infty} n!/n^n = 0$. It is tempting to try and apply the limit theorems for sequences and write down

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{2}{n} \dots \lim_{n\to\infty} \frac{n}{n}.$$

From this expression, one can deduce that since $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ also. In this case the limit is indeed zero, but the logic behind this is flawed, and there are cases where the same reasoning would give the wrong answer.

There are numerous problems with the above expression. First, since the aim of the question is to prove that $n!/n^n$ converges, it is not permissible to write down $\lim_{n\to\infty} n!/n^n$ without first knowing that this actually exists. If the sequence diverged, the $\lim_{n\to\infty} n!/n^n$ may not have meaning. Second, the left hand side has no dependence on n (since the limit has been taken), however the right hand side appears to be dependent on n, being a product of n terms. It is not clear what the right hand side evaluates to.

As a general rule, it is much better to work with expressions for a fixed n, and then take the limit as the final step. For example, it is perfectly permissible to write down, for a fixed n,

$$\frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\dots\left(\frac{n}{n}\right).$$

Here, both sides have a definite meaning. As discussed in the homework solutions, this expression can be manipulated to show that $n!/n^n \leq 1/n$ for all n. After this, taking the limit as $n \to \infty$ shows that $n!/n^n \to 0$.

When can limit theorems be applied?

The limit theorems for adding and multiplying sequences are extremely useful, but it is important to understand exactly when they can be applied. For two sequences (s_n) and (t_n), which are known to converge,

$$\lim_{n\to\infty} s_n t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n.$$

Hence, this theorem tells us that if $x_n \to x$, then $(x_n)^2 \to x^2$. Applying the theorem again, to x_n and x_n^2, shows that $x_n \cdot x_n^2 \to x \cdot x^2$ and hence $x_n^3 \to x^3$. Similarly, for a fixed $l \in \mathbb{N}$, the theorem can be applied repeatedly to show $x_n^l \to x^l$.

However, the theorem will not tell us anything about the sequence $(x_n)^n$. Here, each term in the sequence depends on n, and there is no way to apply the theorem above to learn anything about this sequence.

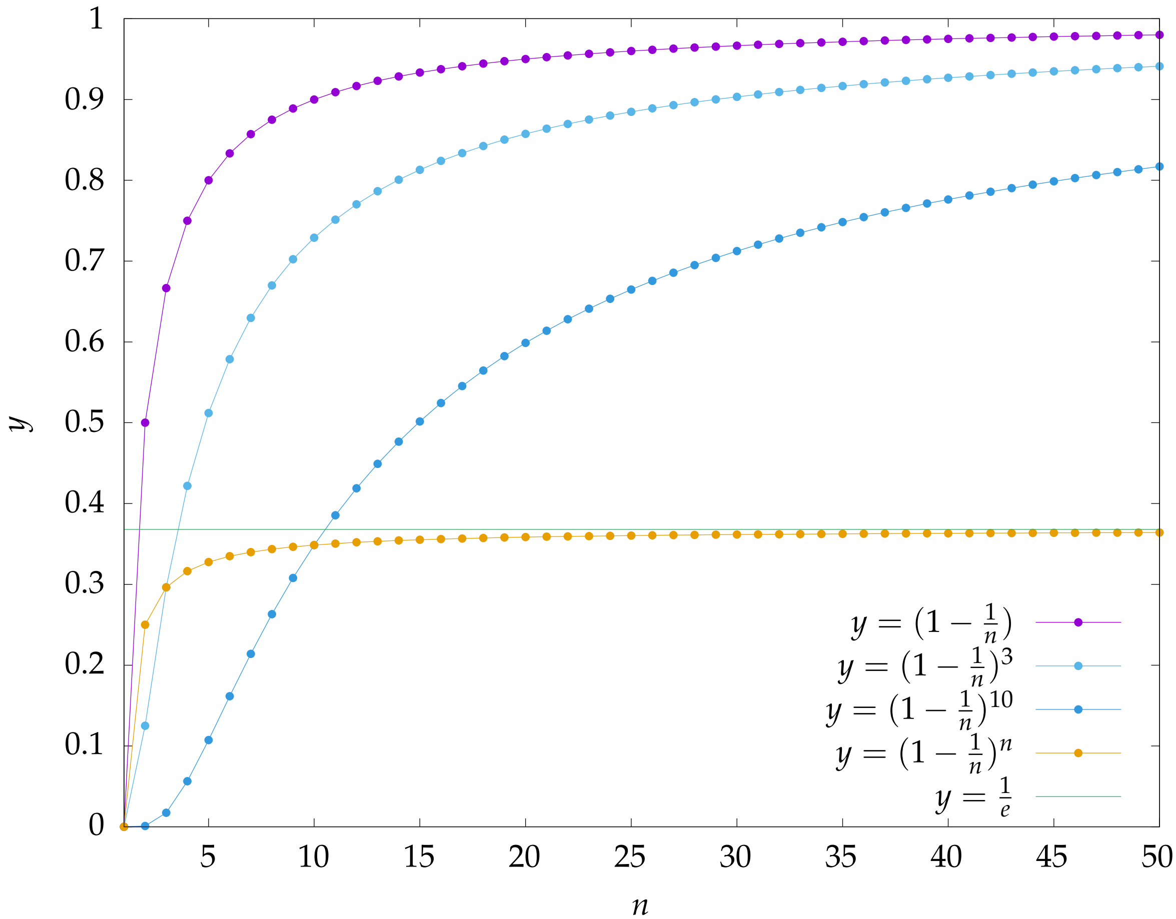

Figure 1: A graph showing different powers of the sequence $(1 - \frac{1}{n})$.

A case where this argument would fail

Suppose $(a_n) = 1 - \frac{1}{n}$. Then $a_n \to 1$ as $n \to \infty$, and by using the result above, for a fixed $l \in \mathbb{N}$, $a_n^l \to 1^l = 1$ as $n \to \infty$. However, consider

$$(a_n)^n = \left(1 - \frac{1}{n}\right)^n.$$

Following the same argument as above, it is tempting to deduce that $(a_n)^n \to 1$ as $n \to \infty$. However, as shown in Fig. 1, this is not the case, and it turns out (although not proved here) that $(a_n)^n \to 1/e$. As l becomes larger, the sequences converge more slowly, but for any fixed l, they will still converge. However, for $(1 - \frac{1}{n})^n$, the effect of the increasing power of n is so strong that it keeps the sequence from converging to 1.