Math 104: Homework 8 (due April 6)

Part 1: Curve sketching

At this point in the class, being able to sketch functions and determine their properties is an important skill, which greatly helps in understanding concepts such as continuity, differentiability, and uniform convergence. Therefore, this week's homework is mainly devoted to this topic.

For questions 1–3, no detailed proofs are required, although you will need to provide some discussion in words about what is going on. To begin, I would like you to try and draw the graphs by hand. There are many ways to do this, such as looking at the behavior as $x \to \pm\infty$, calculating a few specific points and drawing a line through them, using calculus, or searching for zeroes of the function. After this, you can confirm your results using a plotting program. There are many free ones available, such as *Gnuplot* (`www.gnuplot.info`), which runs on Windows, Mac, and Linux.

1. Consider the function

$$f(x) = \begin{cases} 1 - |x-1| & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$$

   defined on the interval $[0, \infty)^\dagger$. Draw $f(x)$.

   (a) Draw $f(x/2)$, $f(x/3)$, and $f(x/4)$, and explain how the shapes of these curves relate to $f(x)$.

   (b) Draw $2f(x)$, $f(x + 1/2)$, $f(x) - 1/2$ and explain how the shapes of these curves relate to $f(x)$.

   (c) Draw $|f(x) - 1/2|$. Is this function continuous? Is it differentiable everywhere?

   (d) Draw $f(x^2)$ and $f(x)^2$.

2. Consider the sequence of functions

$$f_n(x) = \frac{nx^2}{1 + nx^2}$$

   defined on the interval $[0, \infty)$.

   (a) Begin by considering $f_1(x)$. How does it behave as $x \to \infty$? How does it look close to $x = 0$? Use these facts to draw $f_1(x)$.

   (b) Show that $f_n(x) = f_1(\sqrt{n}x)$. By considering question 1(a), use this fact to draw several of the $f_n(x)$.

---

†If you try *Gnuplot*, you can define this function by typing `f(x)=x>2?0:1-abs(x-1)`. It can then be plotted with `plot [0:4] f(x)`.

(c) It can be shown that f_n converges pointwise to a function f defined on $[0, \infty)$ as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Draw $f(x)$ and draw a strip of width $\epsilon = 1/4$ around $f(x)$. If $f_n \to f$ uniformly, then there exists an N such that $n > N$ implies that f_n lies wholly within this strip. Use the graph to explain in words why no such N exists, so that f_n does not converge uniformly to f.

3. Consider the sequence of functions defined on $\mathbb{R}$ as

$$f_n(x) = \begin{cases} x^n \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Draw the sequence of functions $f_0(x)$, $f_1(x)$ and $f_2(x)$. Which of the functions are continuous at $x = 0$? Which of them are differentiable at $x = 0$?

(b) Consider the functions f_n on the interval $[-1/2, 1/2]$, and define $f(x) = 0$. By considering a strip of width ϵ around $f(x)$, explain why f_n will converge uniformly to f on this interval.

4. Consider the function $g_0(x) = |x|$ on $\mathbb{R}$. For $n \in \mathbb{N}$, define $g_n(x) = |g_{n-1}(x) - 2^{1-n}|$.

(a) Draw $g_0(x)$, $g_1(x)$, $g_2(x)$, and $g_3(x)$.

(b) Prove that the functions g_n converge uniformly to a limit g on $\mathbb{R}$.

Part 2: a choice of question

In this part, you can either do question 5 or 6. Question 5 is based on the last midterm problem, and is good practice if you had trouble with this. If you felt confident with the midterm problem, you can try question 6 instead.

5. (a) Consider the function $f(x) = 1 - x$ on $(0, 1)$ and define a sequence of functions as

$$f_n(x) = \begin{cases} 0 & \text{if } x < 1/n, \\ f(x) & \text{if } x \geq 1/n. \end{cases}$$

Sketch $f_1(x)$, $f_2(x)$, and $f_3(x)$, and prove that f_n does not converge uniformly to f.

(b) Now suppose that $f(x) = x - x^2$ instead. Sketch $f_1(x)$, $f_2(x)$, and $f_3(x)$ and prove that in this case f_n converges uniformly to f.

6. Define the sequence of functions on $[0, 1]$ as $h_0(x) = |x - 1/2|$ and $h_n(x) = |h_{n-1}(x) - 3^{-n}|$. Draw several of the h_n. Prove that h_n converges uniformly to a limit h. Where is h differentiable?

Part 3: Additional exercises

7. Ross exercise 28.3
8. Ross exercise 28.8