Math 104: Homework 6 (due March 9)

1. Determine the interior and closure of the following subsets of $\mathbb{R}$:

$$A = \{1/n \,:\, n \in \mathbb{N}\}, \qquad B = [0,1] \cup \mathbb{Q}.$$

2. Consider the function

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

   Show that h is continuous at 0 but at no other point.

3. Ross exercise 17.9

4. In the lectures it was shown that a continuous map from $[0,1]$ to $[0,1]$ has a fixed point. Find an example of a continuous map from $(0,1)$ to $(0,1)$ that does not have a fixed point.

5. Let f be a real-valued function on $\mathbb{R}$. Suppose that for a given $x \in \mathbb{R}$,

$$\lim_{n\to\infty} [f(x + a_n) - f(x - a_n)] = 0$$

   for all sequences (a_n) that converge to 0. Is f continuous at x?

6. Ross exercise 18.9

7. Ross exercise 18.10

8. **Optional for the enthusiasts.** A real-valued function f on an interval I is called convex if for all $x, y \in I$, and $0 < \lambda < 1$, then

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y).$$

   Suppose f is convex on $[a,b]$. Prove that f is continuous at x for $a < x < b$, but need not be continuous at a or b.