Math 104: Homework 5 (due March 2)

1. (a) By using the integral test, or otherwise, prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges if and only if $p > 1$.

(b) **Optional for the enthusiasts.** Suppose (a_n) is a non-increasing sequence of positive real numbers, and that $\sum a_n$ converges. By considering the Cauchy criterion, or otherwise, prove that $na_n \to 0$ as $n \to \infty$. By considering part (a), show that the converse result is not true.

2. Consider the following functions defined for $x, y \in \mathbb{R}$:

$$d_1(x,y) = (x-y)^2, \;\; d_2(x,y) = \sqrt{|x-y|}, \;\; d_3(x,y) = |x^2 - y^2|, \;\; d_4(x,y) = |x - 2y|.$$

For each function, determine whether it is a metric or not.

3. Ross exercise 13.3

4. Consider two-dimensional space $\mathbb{R}^2$, with positions written as $\mathbf{u} = (u_1, u_2)$, and the Euclidean norm defined as $||\mathbf{u}|| = (u_1^2 + u_2^2)^{1/2}$. The Poincaré disk model consists of the points $S = \{\mathbf{u} : ||\mathbf{u}|| < 1\}$, with metric

$$d(\mathbf{u},\mathbf{v}) = \cosh^{-1}\left[1 + \frac{2||\mathbf{u}-\mathbf{v}||^2}{(1-||\mathbf{u}||^2)(1-||\mathbf{v}||^2)}\right]$$

for all $\mathbf{u}, \mathbf{v} \in S$. Define $r = \cosh^{-1} 5/4$. Draw the Poincaré disk, and then calculate and draw the neighborhoods $N_r(\mathbf{u})$ for $\mathbf{u} = (0,0)$, $(1/2, 0)$, and $(3/4, 0)$. *[This can be done analytically, although if you prefer, you can also make use of a computer.]*

5. Suppose that d_1 and d_2 are equivalent metrics for a set S. Prove that if a sequence (s_n) converges to s with respect to d_1, then it also converges with respect to d_2.

6. Suppose that (p_n) is a Cauchy sequence in a set S with metric d, and that some subsequence (p_{n_k}) converges to a point $p \in S$. Prove that the full sequence (p_n) converges to p.

7. Suppose that (p_n) and (q_n) are Cauchy sequences in a set S with metric d. Define $(a_n) = d(p_n, q_n)$. Show that the sequence (a_n) converges. It may be useful to consider the triangle inequality

$$d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

which is true for all n and m.

8. **Optional for the enthusiasts.** Consider two-dimensional space $\mathbb{R}^2$ as in Exercise 4. Define an alternative norm as $||\mathbf{u}||_S = (u_1^2 + u_2^2 + u_1 u_2)^{1/2}$. Prove that the function $d_S(\mathbf{u},\mathbf{v}) = ||\mathbf{u} - \mathbf{v}||_S$ is a metric, and that it is equivalent to the Euclidean metric.