Math 104: Homework 11 (due April 27)

1. Ross exercise 34.3

2. Ross exercise 34.12

3. Ross exercise 36.6

4. Find a sequence of integrable functions (f_n) on $\mathbb{R}$ where $\int_{-\infty}^{\infty} f_n = 1$ for all n, but $f_n \to 0$ uniformly on $\mathbb{R}$.

5. (a) By using simple properties of $\sin x$ and $\cos x$, show how to define the function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. Prove that it is differentiable, strictly increasing, and neither bounded above nor below.

   (b) By using inverse function theorems, define $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ and show that
$$(\tan^{-1})'(x) = \frac{1}{1+x^2}.$$

   (c) Prove that for $|x| < 1$,
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

   (d) By making use of Abel's theorem, or otherwise, show that
$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

   (e) **Optional for the enthusiasts.** Calculate $(5+i)^4(239-i)$ and use it to prove Machin's formula
$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

6. Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$.

   (a) Prove that $I_0 = \frac{\pi}{2}$ and $I_1 = 1$.

   (b) Use integration by parts to prove that $(n+1)I_n = (n+2)I_{n+2}$ for all $n \geq 0$.

   (c) Prove that $I_{2m+1} \leq I_{2m} \leq (1 + \frac{1}{2m})I_{2m+1}$ for all $m \in \mathbb{N}$, and hence that $I_{2m}/I_{2m+1} \to 1$ as $m \to \infty$.

   (d) Prove that for $m \in \mathbb{N}$,
$$\frac{\pi}{2} = \frac{2}{1}\frac{4}{3}\frac{6}{5}\cdots\frac{2m}{2m-1}I_{2m}, \qquad 1 = \frac{3}{2}\frac{5}{4}\frac{7}{6}\cdots\frac{2m+1}{2m}I_{2m+1},$$
   and hence that
$$\frac{\pi}{2} = \lim_{m\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots\frac{2m}{2m-1}\frac{2m}{2m+1}.$$