Math 104: Homework 10 (due April 20)

1. (a) Construct a Taylor series expansion for the function $f(x) = \log(1+x)$ at $x = 0$. You can assume basic properties of logarithm.

   (b) Use Taylor's theorem to write down an expression for the remainder $R_n(x)$, and use this to prove that the Taylor series agrees with f in the range $x \in (-1/2, 1)$. *Note: it can be shown that the Taylor series agrees with f for $x \in (-1, 1)$, as discussed in Chapter 26 of Ross. However, here, to illustrate Taylor's theorem, only a subset of this interval is considered.*

2. Suppose f is a continuous function on $[a, b]$, and $f(x) \geq 0$ for all $x \in [a, b]$. Prove that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

3. Construct an example of a function where $f(x)^2$ is integrable on $[0, 1]$ but $f(x)$ is not.

4. Ross exercise 33.7

5. Ross exercise 33.8

6. (a) For any two numbers $u, v \in \mathbb{R}$, prove that $uv \leq (u^2 + v^2)/2$. Let f and g be two integrable functions on $[a, b]$. Prove that if $\int_a^b f^2 = 1$ and $\int_a^b g^2 = 1$ then

$$\int_a^b fg \leq 1.$$

   (b) Prove the Schwarz inequality, that for any two integrable functions f and g on an interval $[a, b]$,

$$\left| \int_a^b fg \right| \leq \left(\int_a^b f^2 \right)^{1/2} \left(\int_a^b g^2 \right)^{1/2}.$$

   (c) Let X be the set of all continuous functions on the interval $[a, b]$. For any $f, g \in X$, define

$$d(f, g) = \left(\int_a^b |f - g|^2 \right)^{1/2}.$$

   Prove that d is a metric.