Math 104: Final exam solutions

1. Suppose that (s_n) is an increasing sequence with a convergent subsequence. Prove that (s_n) is a convergent sequence.

   Answer: Let the convergent subsequence be (s_{n_k}) that converges to a limit s. Then there exists a K such that $k > K$ implies $|s_{n_k} - s| < 1$, and hence for $k > K$, $s_{n_k} < s + 1$. Suppose that (s_n) is not bounded above. Then there exists an N such that $s_N > s + 1$, and since it is an increasing sequence, $s_n > s + 1$ for all $n > N$. Choosing an n_k such that $k > K$ and $n_k > N$ gives $s_{n_k} < s + 1$ and $s_{n_k} > s + 1$ which is a contradiction. Hence (s_n) is bounded above, and since it is increasing it converges.

2. (a) Let $f_n(x) = \frac{x}{n^2(1+nx^2)}$ be functions defined on $\mathbb{R}$, for all $n \in \mathbb{N}$. By using the Weierstraß M-test, or otherwise, prove that $\sum f_n$ converges uniformly on $\mathbb{R}$.

Answer: If $|x| < 1$, then

$$|f_n(x)| < \left|\frac{1}{n^2(1+nx^2)}\right| \leq \left|\frac{1}{n^2}\right| = \frac{1}{n^2}.$$

If $|x| \geq 1$ then

$$|f_n(x)| \leq \left|\frac{x}{n^2(nx^2)}\right| \leq \left|\frac{1}{n^2(nx)}\right| \leq \frac{1}{n^2}.$$

Hence $f_n(x) < 1/n^2$ for all $x \in \mathbb{R}$. Since $\sum 1/n^2$ converges, it follows that $\sum f_n$ converges by the Weierstraß M-test.

(b) Consider the sequence of functions defined on $\mathbb{R}$ as

$$g_n(x) = \begin{cases} 1/n & \text{for } n-1 \leq x < n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Define $M_n = \sup\{|g_n(x)| : x \in \mathbb{R}\}$. Prove that $\sum g_n$ converges uniformly to a limit g, but that $\sum M_n$ diverges. Sketch $g(x)$ for $0 \leq x < 5$.

Answer: From the definition, it can be seen that $M_n = 1/n$, and hence $\sum M_n$ diverges. Consider $x \in [n-1, n)$ for some $n \in \mathbb{N}$. Then $\sum_{k=1}^{n} f_n(x) = 1/n$. Since $f_k(x) = 0$ for all $k > n$ it follows that $\sum_{k=1}^{\infty} f_n(x) = 1/n$. Thus $\sum g_n$ converges pointwise to

$$g(x) = \begin{cases} 1/m & \text{if } x \in [m-1, m) \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

To show that convergence is uniform, consider any $\epsilon > 0$. By the Archimedean property, there exists an $N \in \mathbb{N}$ such that $1/N < \epsilon$. For $n > N$, note that

$$\left|g(x) - \sum_{k=1}^{n} g_k(x)\right| = \begin{cases} 1/m & \text{if } x \in [m-1, m) \text{ for some } m \in \mathbb{N} \text{ with } m > n, \\ 0 & \text{otherwise,} \end{cases}$$

and thus $|g(x) - \sum_{k=1}^{n} g_k(x)| < 1/n < \epsilon$, so the convergence is uniform. The function g is plotted in Fig. 1.

3. Consider the sequence defined recursively according to $s_1 = 1/3$ and $s_{n+1} = \lambda s_n(1 - s_n)$ for $n \in \mathbb{N}$, where λ is a real constant.

   (a) Prove that (s_n) converges for $\lambda = 1$.

   Answer: Suppose that $s_n \in (0,1)$. Then $s_{n+1} = s_n(1 - s_n) < s_n$ and $s_{n+1} > 0$, so $s_{n+1} \in (0, s_n)$. Since $s_1 = 1/3$ it follows by induction that (s_n) is a decreasing sequence that is bounded below by 0, so it converges.

   (b) Prove that (s_n) has a convergent subsequence for $\lambda = 4$.

   Answer: First, note that

$$s_{n+1} = 4(s_n - s_n^2) = 4\left(\frac{1}{4} - \left(s_n - \frac{1}{2}\right)^2\right) = 1 - 4\left(s_n - \frac{1}{2}\right)^2.$$

   Suppose that $s_n \in [0,1]$. Then $s_n - 1/2 \in [-1/2, 1/2]$, so $4(s_n - 1/2)^2 \in [0,1]$, and hence $s_{n+1} \in [0,1]$. Since $s_1 \in [0,1]$ it follows by mathematical induction that $s_n \in [0,1]$ for all $n \in \mathbb{N}$. Thus the sequence is bounded, so it has a convergent subsequence via the Bolzano–Weierstraß theorem.

   (c) Prove that (s_n) diverges for $\lambda = 12$.

   Answer: Now suppose that for $n \geq 2$, $|s_n| > 2^{n-1}$. Then

$$|s_{n+1}| = 12|s_n| \cdot |1 - s_n| > 12 \cdot 2^{n-1} \cdot (2^{n-1} - 1) > 2 \cdot 2^{n-1} \cdot 1 = 2^n.$$

   so $|s_{n+1}| > 2^{(n+1)-1}$. Since $s_2 = 8/3 > 2$ it follows by mathematical induction that $|s_n| > 2^{n-1}$ for all $n \in \mathbb{N}$ with $n > 2$. Since $2^n \to \infty$ as $n \to \infty$, it follows that s_n is unbounded and hence diverges.

4. A real-valued function f defined on a set S is defined to be *Lipschitz continuous* if there exists a $K > 0$ such that for all $x, y \in S$,

$$|f(x) - f(y)| \leq K|x - y|.$$

(a) Prove that if f is Lipschitz continuous on S, then it is uniformly continuous on S. **Answer:** Choose $\epsilon > 0$. Then set $\delta = \frac{\epsilon}{K}$. If $x, y \in S$ such that $|x - y| < \delta$, then

$$|f(x) - f(y)| \leq K|x - y| < K\delta = \epsilon$$

and hence f is uniformly continuous.

(b) Consider the function $f(x) = \sqrt{x}$ on the interval $[0, \infty)$. Prove that it is uniformly continuous, but not Lipschitz continuous.

Answer: To show that f is uniformly continuous, choose $\epsilon > 0$, and note that

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}}.$$

Suppose that $x < \epsilon^2$ and $y < \epsilon^2$. Then $0 < f(x) < \epsilon$ and $0 < f(y) < \epsilon$ so $|f(x) - f(y)| < \epsilon$. Otherwise, either $x \geq \epsilon^2$ or $y \geq \epsilon^2$, so $\sqrt{x} + \sqrt{y} \geq \epsilon$. If $|x - y| < \delta$ where $\delta = \epsilon^2$, then

$$|f(x) - f(y)| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} < \frac{\epsilon^2}{\epsilon} = \epsilon,$$

and hence the function is uniformly continuous. To see that f is not Lipschitz continuous, consider $x = 1/n^2$, and $y = 0$:

$$\frac{|f(x) - f(y)|}{|x - y|} = \frac{|f(1/n^2)|}{1/n^2} = n.$$

Since n can be made arbitrarily large, there does not exist any K such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in [0, \infty)$.

5. Consider the function defined on the closed interval $[a, b]$ as

$$h_c(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{if } x \neq c, \end{cases}$$

where $a < c < b$.

(a) Prove that h_c is integrable on $[a, b]$ and that $\int_a^b h_c = 0$.

Answer: Define $\eta = \min\{b - c, c - a\}$. For $\alpha < \eta$, define the partition $P_\alpha = \{a = t_0 < t_1 < t_2 < t_3 = b\}$ where $t_1 = c - \alpha$ and $t_2 = c + \alpha$. Then

$$\begin{aligned} L(h_c, P_\alpha) &= \sum_{k=1}^{3} m(h_c, [t_k, t_{k+1}])(t_{k+1} - t_k) \\ &= 0 + 2\alpha m(h_c, [c - \alpha, c + \alpha]) + 0 = 0 \end{aligned}$$

and

$$\begin{aligned} U(h_c, P_\alpha) &= \sum_{k=1}^{3} M(h_c, [t_k, t_{k+1}])(t_{k+1} - t_k) \\ &= 0 + 2\alpha M(h_c, [c - \alpha, c + \alpha]) + 0 = 2\alpha. \end{aligned}$$

Since α can be made arbitrarily small, it follows that $U(h_c) \leq 0$ and $L(h_c) \geq 0$. Hence $U(h_c) = L(h_c) = 0$, and h_c is integrable with integral $\int_a^b h_c = 0$.

(b) Suppose f is integrable on $[a, b]$, and that g is a function on $[a, b]$ such that $f(x) = g(x)$ except at finitely many x in (a, b). Prove that $g(x)$ is integrable and that $\int_a^b f = \int_a^b g$. You can make use of basic properties of integrable functions.

Answer: If g differs from f at finitely many points, then it is possible to write

$$g(x) = f(x) + \sum_{i=1}^{n} b_i h_{c_i}(x)$$

where the c_i are the points where they differ and $b_i = g(c_i) - f(c_i)$. Since g is the sum of integrable functions it is integrable and hence

$$\int_a^b g = \int_a^b f + \sum_{i=1}^{n} \int_a^b b_i h_{c_i} = \int_a^b f.$$

6. Consider the sequence of functions h_n on $\mathbb{R}$ according to

$$h_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n, \\ -n^2 & \text{if } -1/n < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch h_1 and h_2.

Answer: The functions are shown in Fig. 2

(b) Prove that h_n converges pointwise to 0 on $\mathbb{R}$.

Answer: At $x = 0$, $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} 0 = 0$. For $x \neq 0$, there exists an $N \in \mathbb{N}$ such that $1/N < |x|$. Hence for $n > N$, $h_n(x) = 0$ and thus $\lim_{n\to\infty} h_n(x) = 0$.

(c) Let f be a real-valued function on $\mathbb{R}$ that is differentiable at $x = 0$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n f = f'(0).$$

Answer: Consider any $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|x - 0| < \delta$ implies

$$\left| \frac{f(x) - f(0)}{x - 0} - f'(0) \right| = \left| \frac{f(x)}{x} - f'(0) \right| < \epsilon. \tag{1}$$

By the Archimedean property, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. For $n > N$,

$$\int_{-\infty}^{\infty} h_n f = \lim_{c\to-\infty} \int_c^0 h_n f + \lim_{d\to\infty} \int_0^d h_n f = -\int_{-1/n}^0 n^2 f + \int_0^{1/n} n^2 f.$$

By using Eq. 1,

$$\begin{aligned} \int_0^{1/n} f - \int_{-1/n}^0 f &\geq \int_0^{1/n} (f(0) - x(f'(0) - \epsilon))dx - \int_{-1/n}^0 (f(0) - x(f'(0) + \epsilon))dx \\ &= \frac{f'(0) - \epsilon}{2n^2} - \frac{f'(0) - \epsilon}{2n^2} = \frac{f'(0)}{n^2} - \frac{\epsilon}{n^2}. \end{aligned}$$

By the same procedure

$$\int_0^{1/n} f - \int_{-1/n}^0 f \leq \frac{f'(0)}{n^2} + \frac{\epsilon}{n^2}$$

and hence

$$\left| \int_{-\infty}^{\infty} h_n f - f'(0) \right| = \left| n^2 \left(\int_0^{1/n} f - \int_{-1/n}^0 f \right) - f'(0) \right| \leq \epsilon$$

from which it follows that $\lim_{n\to\infty} h_n f = f'(0)$.

7. (a) Find an example of a set $A \subseteq \mathbb{R}$ where the interior A° is non-empty, but that $\sup A \neq \sup A^\circ$ and $\inf A \neq \inf A^\circ$.

Answer: Consider $A = \{-2\} \cup (-1,1) \cup \{2\}$. Then $\sup A = 2$ and $\inf A = -2$. However, $A^\circ = (-1,1)$, since if $x = \pm 2$, then there is no neighborhood $N_\delta(x) \subseteq A$ for any $\delta > 0$. Hence $\inf A^\circ = -1$ and $\sup A^\circ = 1$.

(b) Let $B_1, \ldots, B_n$ be subsets of $\mathbb{R}$. Prove that

$$\left(\bigcap_{i=1}^{n} B_i\right)^\circ = \bigcap_{i=1}^{n} B_i^\circ.$$

Answer: Call the LHS E and the RHS D. Suppose $x \in E$. Then there exists a $\delta > 0$ such that the neighborhood $N_\delta(x) \subseteq \bigcap_{i=1}^{n} B_i$. Hence $N_\delta(x) \subseteq B_i$ for all $i = 1, \ldots, n$ and so $x \in B_i^\circ$ for all $i = 1, \ldots, n$. Hence $x \in D$.

Now suppose that $x \in D$. Then $x \in B_i^\circ$ for all $i = 1, \ldots, n$, so there exist $\delta_i > 0$ such that $N_{\delta_i}(x) \subseteq B_i$ for all $i = 1, \ldots, n$. Let $\delta = \min\{\delta_1, \ldots, \delta_n\} > 0$. Then $N_\delta(x) \subseteq B_i$ for all $i = 1, \ldots, n$ and hence $N_\delta(x) \subseteq \bigcap_{i=1}^{n} B_i$, so $x \in E$.

(c) Suppose that $\{C_i\}_{i=1}^{\infty}$ is an infinite sequence of subsets of $\mathbb{R}$. Prove that

$$\left(\bigcap_{i=1}^{n} C_i\right)^\circ \subseteq \bigcap_{i=1}^{n} C_i^\circ$$

but that these two sets may not be equal.

Answer: The first argument from part (b) can be still be applied when there are an infinite number of sets, to establish that $(\bigcap_{i=1}^{n} C_i)^\circ \subseteq \bigcap_{i=1}^{n} C_i^\circ$. However, the second argument may fail if the infimum of the δ_i is equal to zero. Motivitated by this, consider $C_i = (-1/i, 1/i)$. Since these are open sets, $C_i^\circ = C_i$. Hence,

$$\bigcap_{i=1}^{\infty} C_i^\circ = \{0\}.$$

However

$$\bigcap_{i=1}^{\infty} C_i = \{0\}$$

and thus

$$\left(\bigcap_{i=1}^{\infty} C_i\right)^\circ = \emptyset.$$

8. (a) If f is a continuous strictly increasing function on $\mathbb{R}$, prove that

$$d(x,y) = |f(x) - f(y)|$$

defines a metric on $\mathbb{R}$.

Answer: Consider the three properties of being a metric

M1. For all $x \in \mathbb{R}$, $d(x,x) = |f(x) - f(x)| = 0$. If $d(x,y) = 0$ then $f(x) = f(y)$, and if f is strictly increasing then $x = y$.

M2. For all $x, y \in \mathbb{R}$, $d(x,y) = |f(x) - f(y)| = |f(y) - f(x)| = d(y,x)$ and thus the metric is symmetric.

M3. For all $x, y, z \in \mathbb{R}$,

$$d(x,y) + d(y,z) = |f(x) - f(y)| + |f(y) - f(z)| \leq |f(x) - f(z)| = d(x,z)$$

which follows from the usual triangle inequality.

(b) Prove that d is equivalent to the Euclidean metric $d_E(x,y) = |x - y|$.

Answer: Write $N_r(x)$ and $N_r^E(x)$ for the neighborhoods of radius r with respect to d and d_E respectively. To prove that the two metrics are equivalent, consider any $x \in \mathbb{R}$, and $\epsilon > 0$. By continuity, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Hence $N_\delta^E(x) \subseteq N_\epsilon(x)$.

Since f is continuous and strictly increasing, it has a continous strictly increasing inverse f^{-1}. Consider any $x \in \mathbb{R}$ and $\epsilon > 0$. By using continuity of f^{-1} at $f(x)$, there exists a $\delta > 0$ such that $|f(x) - z| < \delta$ implies $|f^{-1}(f(x)) - f^{-1}(z)| < \epsilon$ so $|x - f^{-1}(z)| < \epsilon$. Then

$$\begin{aligned} N_\delta(x) &= \{y \in \mathbb{R} : |f(x) - f(y)| < \delta\} \\ &\subseteq \{y \in \mathbb{R} : |x - f^{-1}(f(y))| < \epsilon\} \\ &= \{y \in \mathbb{R} : |x - y| < \epsilon\} = N_\epsilon^E(x). \end{aligned}$$

Hence the two metrics are equivalent.

(c) Suppose g is a continuous strictly increasing function on $[0, \infty)$ where $g(0) = 0$. Is the function

$$d_2(x,y) = g(|x - y|)$$

always a metric? Either prove the result, or find a counterexample.

Answer: This does not always define a metric. Consider $g(x) = x^2$. Then $d_2(0,1) + d_2(1,2) = 1^2 + 1^2 = 2$ but $d_2(0,2) = 2^2 = 4$ so the triangle inequality is violated.

9. Consider the continuous function defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{x}{e^x - 1} & \text{if } x \neq 0, \\ c & \text{if } x = 0, \end{cases}$$

where $c \in \mathbb{R}$. For this question you can assume basic properties of the exponential, such that it is continuous, differentiable, and has the Taylor series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(a) Use L'Hôpital's rule to compute $\lim_{x\to 0} f(x)$ and hence determine c.

Answer: By L'Hopital's rule, $\lim_{x\to 0} f(x) = \lim_{x\to 0} {}^{1}/_{e^x} = 1$. A function is continuous at 0 if and only if $\lim_{x\to 0} f(x) = f(0)$, and thus $c = 1$.

(b) Show that f is differentiable on $\mathbb{R}$ and compute f'.

Answer: The derivative for $x \neq 0$ is

$$f'(x) = \frac{(e^x - 1) - xe^x}{(e^x - 1)^2}.$$

Using the definition of the derivative,

$$\begin{aligned} f'(0) = \lim_{a\to 0} \frac{f(a) - f(0)}{a - 0} &= \lim_{a\to 0} \frac{\frac{a}{e^a - 1} - 1}{a} \\ &= \lim_{a\to 0} \frac{a - e^a + 1}{a(e^a - 1)} \\ &= \lim_{a\to 0} \frac{1 - e^a}{(a+1)e^a - 1} \\ &= \lim_{a\to 0} \frac{-e^a}{e^a(a+2)} = -\frac{1}{2} \end{aligned}$$

where L'Hôpital's rule has been applied twice.

(c) Calculate the function limits

$$\lim_{x\to\infty} f(x), \qquad \lim_{x\to-\infty} (x + f(x)).$$

Use the results to sketch f and f_T on $(-10, 10)$.

Answer: The first limit is

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{\frac{e^x}{x} - \frac{1}{x}} = 0$$

since $e^x/x \to \infty$ as $x \to \infty$; this can be verified by using the Taylor series to see that $e^x > x^2/2$ for all $x > 0$. The second limit is

$$\lim_{x\to-\infty} (x + f(x)) = \lim_{x\to-\infty} x\frac{x(e^x - 1 + 1)}{e^x - 1} = \lim_{x\to\infty} \frac{xe^x}{e^x - 1} = 0$$

since as $xe^x \to 0$ as $x \to -\infty$. The functions f and f_T are plotted in Fig. 3.

10. Given a function f on $[a,b]$, define the *total variation* of f to be

$$Vf = \sup\left\{\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|\right\}$$

where the supremum is taken over all partitions $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a,b]$.

(a) Calculate Vf for the function defined on $[-1,1]$ as

$$f(x) = \begin{cases} -2 & \text{if } x < 0, \\ 3 & \text{if } x \geq 0. \end{cases}$$

Answer: Consider any partition $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$. Then there exists a k such that $t_{k-1} < 0 \leq t_k$. For $j < k$, $|f(t_j) - f(t_{j-1}| = |(-2) - (-2)| = 0$. Similarly, if $j > k$, $|f(t_j) - f(t_{j-1}| = |3 - 3| = 0$, and thus

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| = 0 + |f(t_k) - f(t_{k-1})| + 0 = |3 - (-2)| = 5.$$

Since this is true for an arbitrary partition, it follows that $Vf = 5$.

(b) Prove that if f is differentiable on an interval $[a,b]$, and that f' is continuous then $Vf = \int_a^b |f'|$.

Answer: To prove that $Vf \leq \int_a^b |f'|$, consider any partition P of $[a,b]$. Then, by using the Fundamental Theorem of Calculus,

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| = \sum_{k=1}^{n} \left| \int_{t_{k-1}}^{t_k} f' \right| \leq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |f'| = \int_a^b |f'|.$$

Since $|f'|$ is integrable on $[a,b]$, for all $\epsilon > 0$, there exists a partition P such that $U(|f'|,P) - L(|f'|,P) < \epsilon$, and hence $L(f',P) > (\int_a^b |f'|) - \epsilon$. By using the Mean Value Theorem, for each k, there exists an $x_k \in (t_{k-1}, t_k)$ such that

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}.$$

Hence

$$\begin{aligned} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| &= \sum_{k=1}^{n} |f'(x_k)(t_k - t_{k-1})| \\ &\geq \sum_{k=1}^{n} m(|f'|, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= L(|f'|, P) > \left(\int_a^b |f'|\right) - \epsilon. \end{aligned}$$

It is therefore possible to find partition such that the sum is larger than $\left(\int_a^b |f'|\right) - \epsilon$ for all $\epsilon > 0$ and thus $Vf \geq \int_a^b |f'|$. Combining this with the first result shows $Vf = \int_a^b |f'|$.

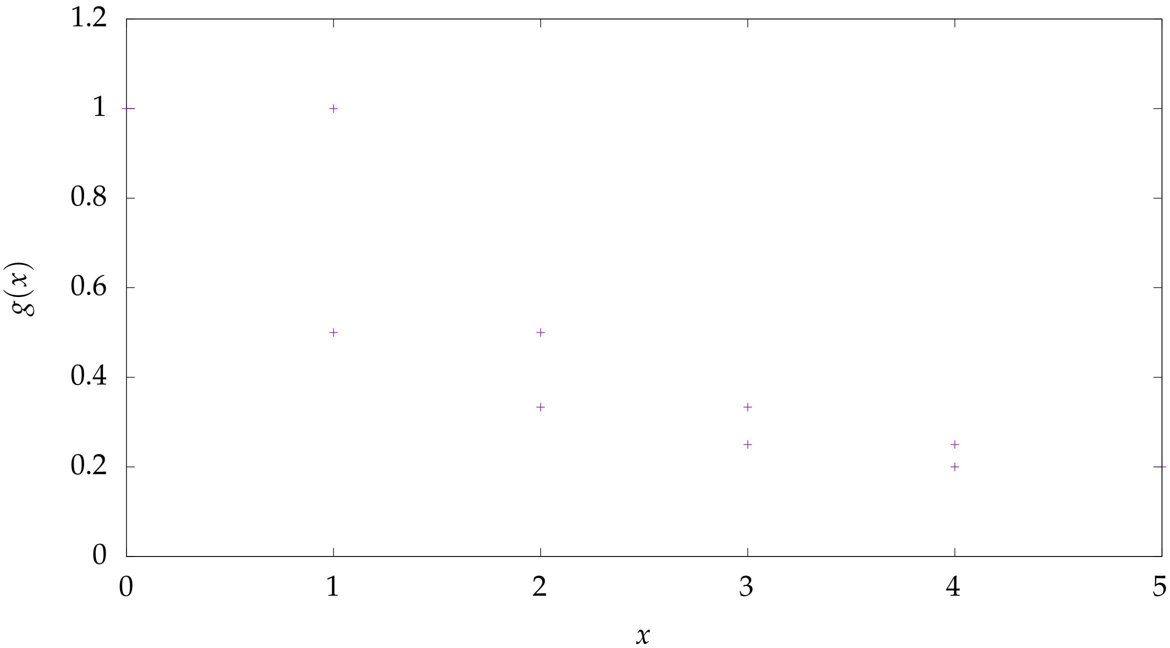

Figure 1: Graph of the function considered in question 2.

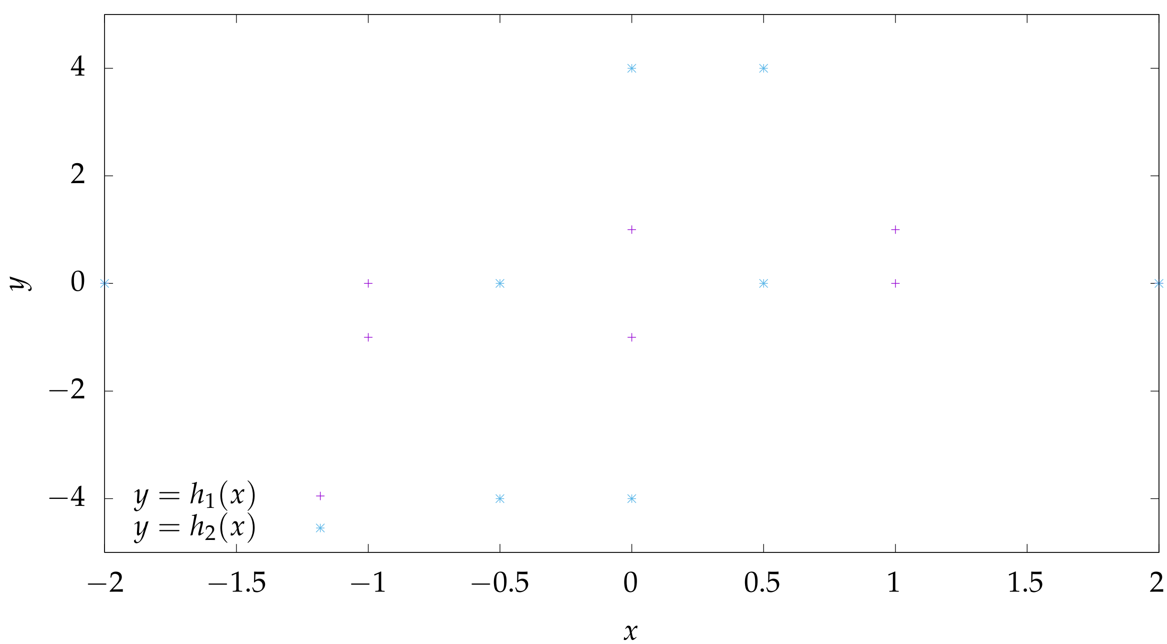

Figure 2: Graph showing the functions h_1 and h_2 considered in question 5.

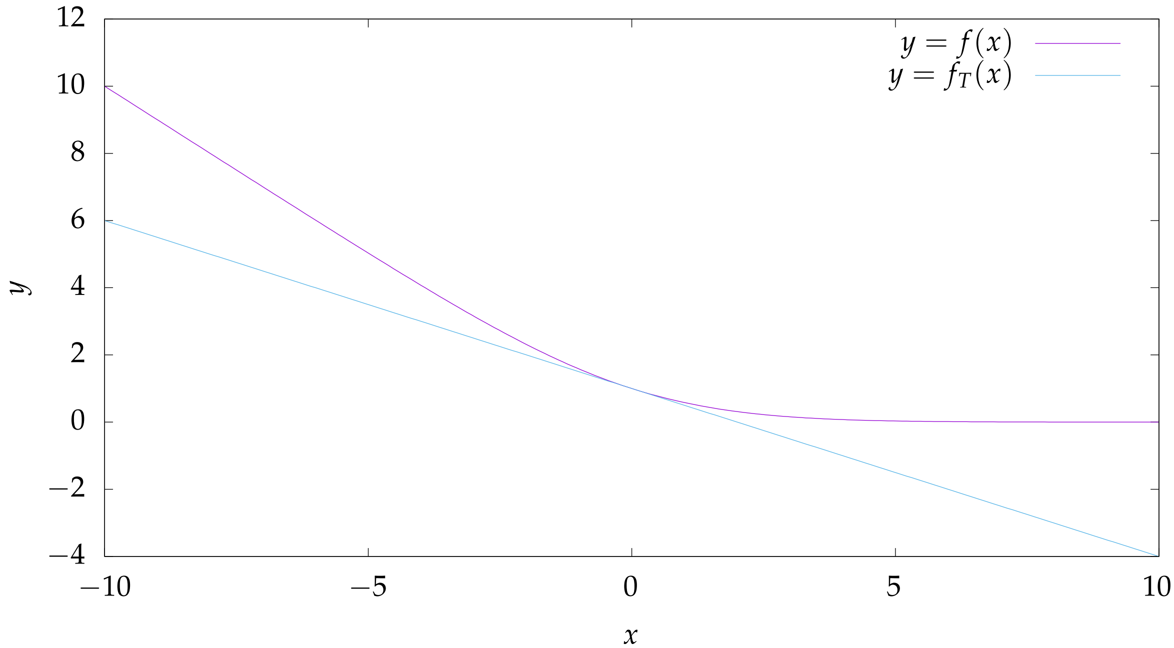

Figure 3: Graph showing the functions considered in question 9 on L'Hôpital's rule.