Math 104: Final exam

1. Suppose that (s_n) is an increasing sequence with a convergent subsequence. Prove that (s_n) is a convergent sequence.

2. (a) Let f_n be functions defined on $\mathbb{R}$ as

$$f_n(x) = \frac{x}{n^2(1+nx^2)},$$

   for all $n \in \mathbb{N}$. By using the Weierstrass M-test, or otherwise, prove that $\sum f_n$ converges uniformly on $\mathbb{R}$.

   (b) Consider the sequence of functions defined on $\mathbb{R}$ as

$$g_n(x) = \begin{cases} 1/n & \text{for } n-1 \leq x < n, \\ 0 & \text{otherwise,} \end{cases}$$

   for all $n \in \mathbb{N}$. Define $M_n = \sup\{|g_n(x)| : x \in \mathbb{R}\}$. Prove that $\sum g_n$ converges uniformly to a limit g, but that $\sum M_n$ diverges. Sketch $g(x)$ for $0 \leq x < 5$.

3. Consider the sequence (s_n) defined according to $s_1 = 1/3$ and $s_{n+1} = \lambda s_n(1 - s_n)$ for $n \in \mathbb{N}$, where λ is a real constant.

   (a) Prove that (s_n) converges for $\lambda = 1$.

   (b) Prove that (s_n) has a convergent subsequence for $\lambda = 4$.

   (c) Prove that (s_n) diverges for $\lambda = 12$.

4. A real-valued function f on a set S is defined to be *Lipschitz continuous* if there exists a $K > 0$ such that for all $x, y \in S$,

$$|f(x) - f(y)| \leq K|x - y|.$$

   (a) Prove that if f is Lipschitz continuous on S, then it is uniformly continuous on S.

   (b) Consider the function $f(x) = \sqrt{x}$ on the interval $[0, \infty)$. Prove that it is uniformly continuous, but not Lipschitz continuous.

5. Consider the function defined on the closed interval $[a, b]$ as

$$h_c(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{if } x \neq c, \end{cases}$$

   where $a < c < b$.

   (a) Prove that h_c is integrable on $[a, b]$ and that $\int_a^b h_c = 0$.

(b) Suppose f is integrable on $[a,b]$, and that g is a function on $[a,b]$ such that $f(x) = g(x)$ except at finitely many x in (a,b). Prove that $g(x)$ is integrable and that $\int_a^b f = \int_a^b g$. You can make use of basic properties of integrable functions.

6. Consider the sequence of functions h_n on $\mathbb{R}$ defined as

$$h_n(x) = \begin{cases} n^2 & \text{if } 0 < x < 1/n, \\ -n^2 & \text{if } -1/n < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch h_1 and h_2.

(b) Prove that h_n converges pointwise to 0 on $\mathbb{R}$.

(c) Let f be a real-valued function on $\mathbb{R}$ that is differentiable at $x = 0$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n f = f'(0).$$

7. (a) Find an example of a set $A \subseteq \mathbb{R}$ where the interior A° is non-empty, but that $\sup A \neq \sup A^\circ$ and $\inf A \neq \inf A^\circ$.

(b) Let $B_1, \ldots, B_n$ be subsets of $\mathbb{R}$. Prove that

$$\left(\bigcap_{i=1}^{n} B_i\right)^\circ = \bigcap_{i=1}^{n} B_i^\circ.$$

(c) Suppose that $\{C_i\}_{i=1}^\infty$ is an infinite sequence of subsets of $\mathbb{R}$. Prove that

$$\left(\bigcap_{i=1}^{\infty} C_i\right)^\circ \subseteq \bigcap_{i=1}^{\infty} C_i^\circ$$

but that these two sets may not be equal.

8. (a) If f is a continuous strictly increasing function on $\mathbb{R}$, prove that

$$d(x,y) = |f(x) - f(y)|$$

defines a metric on $\mathbb{R}$.

(b) Prove that d is equivalent to the Euclidean metric $d_E(x,y) = |x-y|$.

(c) Suppose g is a continuous strictly increasing function on $[0,\infty)$ where $g(0) = 0$. Is the function

$$d_2(x,y) = g(|x-y|)$$

always a metric? Either prove the result, or find a counterexample.

9. Consider the continuous function defined on $\mathbb{R}$ as

$$f(x) = \begin{cases} \frac{x}{e^x - 1} & \text{if } x \neq 0, \\ c & \text{if } x = 0, \end{cases}$$

where $c \in \mathbb{R}$. For this question you can assume basic properties of the exponential, such that it is continuous, differentiable, and has the Taylor series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(a) Use L'Hôpital's rule to compute $\lim_{x \to 0} f(x)$ and hence determine c.

(b) Show that f is differentiable on $\mathbb{R}$ and compute f'.

(c) Use the above results to write down the partial Taylor series

$$f_T(x) = \sum_{k=0}^{1} f^{(k)}(0) \frac{x^k}{k!}.$$

Calculate the function limits $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty}(x + f(x))$ and use the results to sketch f and f_T on $(-10, 10)$.

10. Given a function f on $[a, b]$, define the *total variation* of f to be

$$Vf = \sup \left\{ \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| \right\}$$

where the supremum is taken over all partitions $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$.

(a) Calculate Vf for the function defined on $[-1, 1]$ as

$$f(x) = \begin{cases} -2 & \text{if } x < 0, \\ 3 & \text{if } x \geq 0. \end{cases}$$

(b) Prove that if f is differentiable on an interval $[a, b]$ and that f' is continuous then $Vf = \int_a^b |f'|$.